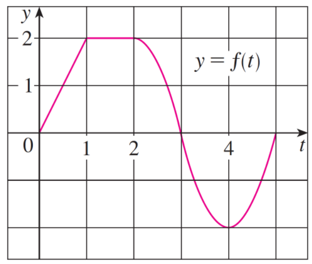
The Fundamental Theorem of Calculus

EXAMPLE: If f is a function whose graph is shown below and $g(x) = \int\limits_0^x f(t)dt$, find the values of $g(0),\ g(1),\ g(2),\ g(3),\ g(4)$, and $g(5)$. Then sketch a rough graph of g.



Solution: First we notice that $g(0) = \int\limits_0^0 f(t)dt = 0$. From the figure above we see that $g(1)$ is the area of a triangle:

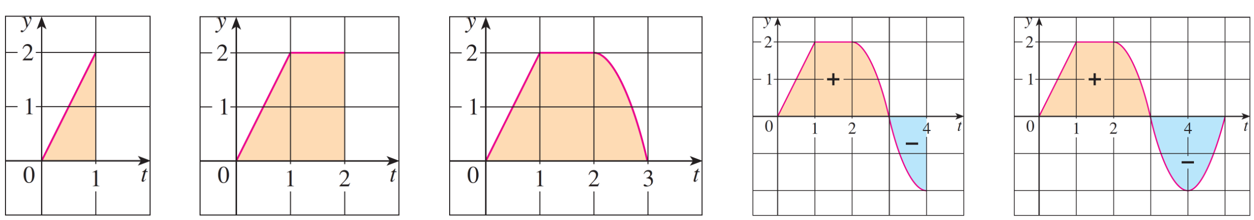
$$g(1) = \int\limits_0^1 f(t)dt = \frac{1}{2}(1 \cdot 2) = 1$$

To find $g(2)$ we add to $g(1)$ the area of a rectangle:

$$g(2) = \int\limits_0^2 f(t)dt = \int\limits_0^1 f(t)dt + \int\limits_1^2 f(t)dt = 1 + (1 \cdot 2) = 3$$

We estimate that the area under f from 2 to 3 in about 1.3, so

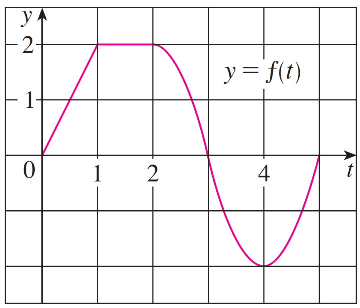
$$g(3) = g(2) + \int\limits_2^3 f(t)dt \approx 3 + 1.3 = 4.3$$



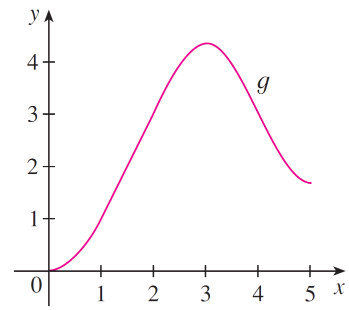
For $t > 3,\ f(t)$ is negative and so we start subtracting areas:

$$g(4) = g(3) + \int\limits_3^4 f(t)dt \approx 4.3 + (-1.3) = 3 \qquad g(5) = g(4) + \int\limits_4^5 f(t)dt \approx 3 + (-1.3) = 1.7$$

We use these values to sketch the graph of g:



$g(0) = 0$
$g(1) = 1$
$g(2) = 3$
$g(3) \approx 4.3$
$g(4) \approx 3$
$g(5) \approx 1.7$



EXAMPLE: If $g(x) = \int\limits_a^x f(t)dt$, where $a = 0$ and $f(t) = \sin t$, find a formula for $g(x)$ and calculate $g'(x)$.

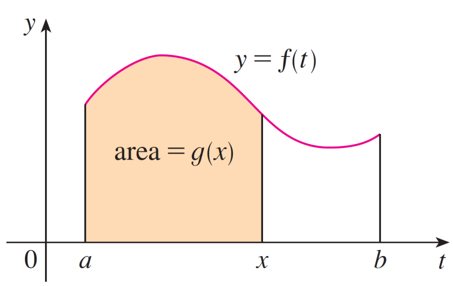
Solution: By the Evaluation Theorem we have:

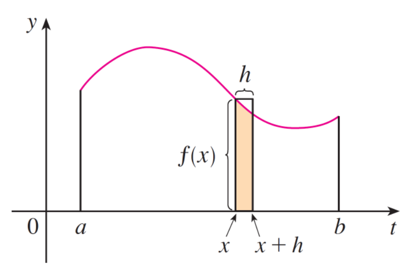
$$g(x) = \int\limits_0^x \sin t dt = -\cos t]_0^x = -\cos x - (-\cos 0) = -\cos x + 1$$

Then $g'(x) = \dfrac{d}{dx}(-\cos x + 1) = \sin x$.

REMARK: To see why this might be generally true we consider a continuous function f with $f(x) \geq 0$. Then

$$g(x) = \int\limits_a^x f(t)dt$$





To compute $g'(x)$ from the definition of derivative we first observe that, for $h > 0$, $g(x+h)-g(x)$ is obtained by subtracting areas, so it is the area under the graph of f from x to $x+h$ (the gold area). For small h you can see that this area is approximately equal to the area of the rectangle with height $f(x)$ and width h:

$$g(x+h) - g(x) \approx hf(x) \quad \Longrightarrow \quad \frac{g(x+h) - g(x)}{h} \approx f(x)$$

Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

THEOREM (The Fundamental Theorem Of Calculus, Part I): If f is continuous on $[a,b]$, then the function g defined by

$$\boxed{g(x)=\int_a^x f(t)dt \qquad a\le x\le b}$$

is an antiderivative of f, that is $\boxed{g'(x)=f(x)}$ for $a<x<b$.

Proof: If x and $x+h$ are in the open interval (a,b), then

$$g(x+h)-g(x)=\int_a^{x+h} f(t)dt-\int_a^x f(t)dt=\left(\int_a^x f(t)dt+\int_x^{x+h} f(t)dt\right)-\int_a^x f(t)dt=\int_x^{x+h} f(t)dt$$

and so, for $h\ne 0$,

$$\frac{g(x+h)-g(x)}{h}=\frac{1}{h}\int_x^{x+h} f(t)dt \tag{1}$$

For now let's assume that $h>0$. Since f is continuous on $[x,x+h]$, the Extreme Value Theorem says that there are numbers u and v in $[x,x+h]$ such that $f(u)=m$ and $f(v)=M$, where m and M are the absolute minimum and maximum values of f on $[x,x+h]$. By Property 8 of integrals $\left(m(b-a)\le\int_a^b f(x)dx\le M(b-a)\right)$ we have

$$mh\le\int_x^{x+h} f(t)dt\le Mh \quad\Longrightarrow\quad f(u)h\le\int_x^{x+h} f(t)dt\le f(v)h$$

Since $h>0$, we can divide this inequality by h:

$$f(u)\le\frac{1}{h}\int_x^{x+h} f(t)dt\le f(v)$$

Now we use (1) to replace the middle part of this inequality:

$$f(u)\le\frac{g(x+h)-g(x)}{h}\le f(v) \tag{2}$$

Inequality (2) can be proved in a similar manner for the case $h<0$. Now let $h\to 0$. Then $u\to x$ and $v\to x$, since u and v lie between x and $x+h$. Thus

$$\lim_{h\to 0} f(u)=\lim_{u\to x} f(u)=f(x) \quad\text{and}\quad \lim_{h\to 0} f(v)=\lim_{v\to x} f(v)=f(x)$$

because f is continuous at x. We conclude, from (2) and the Squeeze Theorem, that

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=f(x) \tag{3}$$

If $x=a$ or b, then (3) can be interpreted as a one-sided limit. We know that *if f is differentiable at a, then f is continuous at a.* If we adopt this theorem for one-sided limits, we obtain that g is continuous on $[a,b]$. ■

EXAMPLE: Find the derivative of the function $g(x)=\int\limits_3^x t^4 dt$.

Solution: Since $f(t)=t^4$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x)=x^4$$

EXAMPLE: Find the derivative of $g(x)=\int\limits_{-1}^x e^{t^2} dt$.

Solution: Since $f(t)=e^{t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x)=e^{x^2}$$

EXAMPLE: Find the derivative of $g(x)=\int\limits_{-1}^{x^2} e^{t^2} dt$.

Solution: Since $f(t)=e^{t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$\frac{d}{dx}\int\limits_{-1}^{x^2} e^{t^2} dt=\frac{d}{dx}\int\limits_{-1}^{u} e^{t^2} dt=\frac{d}{du}\left[\int\limits_{-1}^{u} e^{t^2} dt\right]\frac{du}{dx}=e^{u^2}\frac{du}{dx}=e^{(x^2)^2}\cdot 2x=2xe^{x^4}$$

In short,

$$\frac{d}{dx}\int\limits_{-1}^{x^2} e^{t^2} dt=e^{(x^2)^2}\cdot (x^2)'=2xe^{x^4}$$

EXAMPLE: Find the derivative of $g(x)=\int\limits_2^{\sin x} \sqrt[3]{t} dt$.

EXAMPLE: Find the derivative of $g(x)=\int\limits_{x^4}^{1} \sec t dt$.

EXAMPLE: Find the derivative of $g(x)=\int\limits_{2}^{\sin x}\sqrt[3]{t}dt$.

Solution: Part 1 of the Fundamental Theorem of Calculus gives

$$\frac{d}{dx}\int\limits_{2}^{\sin x}\sqrt[3]{t}dt=\frac{d}{dx}\int\limits_{2}^{u}\sqrt[3]{t}dt=\frac{d}{du}\left[\int\limits_{2}^{u}\sqrt[3]{t}dt\right]\frac{du}{dx}=\sqrt[3]{u}\frac{du}{dx}=\sqrt[3]{\sin x}\cos x$$

In short,

$$\frac{d}{dx}\int\limits_{2}^{\sin x}\sqrt[3]{t}dt=\sqrt[3]{\sin x}(\sin x)'=\sqrt[3]{\sin x}\cos x$$

EXAMPLE: Find the derivative of $g(x)=\int\limits_{x^4}^{1}\sec t dt$.

Solution: By Property 1 of the Definite Integral and Part 1 of the Fundamental Theorem of Calculus we have

$$\begin{aligned}\frac{d}{dx}\int\limits_{x^4}^{1}\sec t dt=-\frac{d}{dx}\int\limits_{1}^{x^4}\sec t dt=-\frac{d}{dx}\int\limits_{1}^{u}\sec t dt=-\frac{d}{du}\left[\int\limits_{1}^{u}\sec t dt\right]\frac{du}{dx}&=-\sec u\frac{du}{dx}\\&=-\sec(x^4)\cdot 4x^3\\&=-4x^3\sec(x^4)\end{aligned}$$

In short,

$$\frac{d}{dx}\int\limits_{x^4}^{1}\sec t dt=-\frac{d}{dx}\int\limits_{1}^{x^4}\sec t dt=-\sec(x^4)(x^4)'=-4x^3\sec(x^4)$$

EXAMPLE: Find the derivative of $g(x)=\int\limits_{\sin x}^{\cos x}t^2dt$.

EXAMPLE: Find the derivative of $g(x) = \int\limits_{\sin x}^{\cos x} t^2 dt$.

Solution: We have

$$\int\limits_{\sin x}^{\cos x} t^2 dt = \int\limits_{\sin x}^{0} t^2 dt + \int\limits_{0}^{\cos x} t^2 dt = -\int\limits_{0}^{\sin x} t^2 dt + \int\limits_{0}^{\cos x} t^2 dt$$

therefore

$$\begin{aligned}\frac{d}{dx}\int\limits_{\sin x}^{\cos x} t^2 dt &= -\frac{d}{dx}\int\limits_{0}^{\sin x} t^2 dt + \frac{d}{dx}\int\limits_{0}^{\cos x} t^2 dt \\ &= -\sin^2 x(\sin x)' + \cos^2 x(\cos x)' \\ &= -\sin^2 x\cos x + \cos^2 x(-\sin x) \\ &= -\sin^2 x\cos x - \cos^2 x\sin x\end{aligned}$$

THEOREM (The Fundamental Theorem Of Calculus, Part II): If f is continuous on $[a, b]$, then

$$\boxed{\int\limits_a^b f(x)dx = F(b) - F(a)}$$

where F is any antiderivative of f, that is $F' = f$.

Proof: Put

$$g(x) = \int\limits_a^x f(t)dt$$

By the Fundamental Theorem Of Calculus, Part I, $g(x)$ is an antiderivative of $f(x)$. Therefore any other antiderivative $F(x)$ of $f(x)$ can be written as

$$F(x) = g(x) + C = \int\limits_a^x f(t)dt + C$$

It follows that

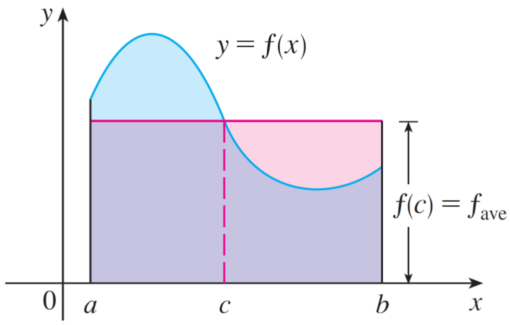
$$F(a) = \int\limits_a^a f(t)dt + C = 0 + C = C \quad \Longrightarrow \quad F(b) = \int\limits_a^b f(t)dt + C = \int\limits_a^b f(t)dt + F(a)$$

thus

$$F(b) = \int\limits_a^b f(t)dt + F(a) \quad \Longrightarrow \quad F(b) - F(a) = \int\limits_a^b f(t)dt \quad \blacksquare$$

THEOREM (The Mean Value Theorem for Integrals): If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that

$$\boxed{f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx} \quad \text{or} \quad \boxed{\int_a^b f(x)dx = f(c)(b-a)}$$



EXAMPLE: Find the average value of the function $f(x) = \sqrt{x}$ on the interval $[1,4]$.

Solution: We have

$$f_{ave} = \frac{1}{4-1}\int_1^4 \sqrt{x}dx = \frac{1}{3}\int_1^4 x^{1/2}dx = \frac{1}{3}\cdot\frac{x^{1/2+1}}{1/2+1}\Bigg]_1^4 = \frac{1}{3}\cdot\frac{2}{3}x^{3/2}\Bigg]_1^4 = \frac{1}{3}\left(\frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2}\right) = \frac{14}{9}$$

We now find c:

$$f(c) = f_{ave} = \frac{14}{9} \quad\Longrightarrow\quad \sqrt{c} = \frac{14}{9} \quad\Longrightarrow\quad c = \left(\frac{14}{9}\right)^2 = \frac{14^2}{9^2} = \frac{196}{81}$$

EXAMPLE: Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1,2]$.

EXAMPLE: Find the average value of the function $f(x) = \sqrt{4-x^2}$ on the interval $[-2,2]$.

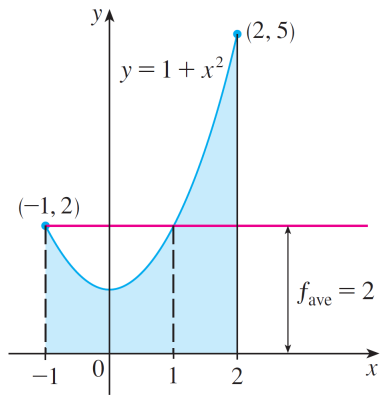
EXAMPLE: Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Solution: We have

$$f_{ave} = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)dx = \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = 2$$

We now find c:

$$f(c) = f_{ave} = 2 \quad \Longrightarrow \quad 1 + c^2 = 2 \quad \Longrightarrow \quad c^2 = 1 \quad \Longrightarrow \quad c = \pm 1$$



EXAMPLE: Find the average value of the function $f(x) = \sqrt{4 - x^2}$ on the interval $[-2, 2]$.

Solution: We have

$$f_{ave} = \frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}dx = \frac{1}{4}\cdot\frac{\pi\cdot 2^2}{2} = \frac{\pi}{2}$$

We now find c:

$$f(c) = f_{ave} = \frac{\pi}{2} \quad \Longrightarrow \quad \sqrt{4-c^2} = \frac{\pi}{2} \quad \Longrightarrow \quad 4 - c^2 = \frac{\pi^2}{4} \quad \Longrightarrow \quad c = \pm\sqrt{4 - \frac{\pi^2}{4}} \approx \pm 1.23798$$