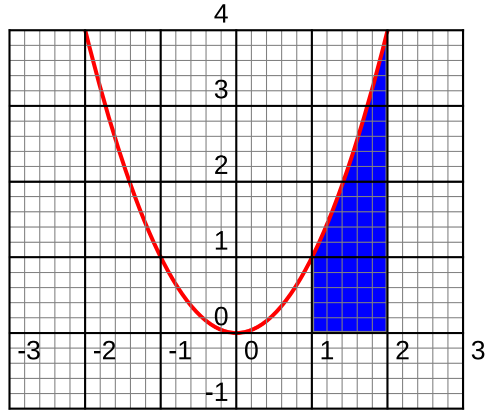
Evaluating Definite Integrals

THEOREM (Evaluation Theorem): Suppose f is integrable on $[a,b]$ and $f = F'$ for an other function F. Then

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big]_a^b$$

EXAMPLES:



1. $$\int_1^2 x^2 dx = \frac{x^3}{3}\Big]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

2. $$\int_0^{\pi/2} \sin x dx = -\cos x\Big]_0^{\pi/2} = -\cos(\pi/2) - (-\cos(0))$$
$$= 0 - (-1) = 1$$

3. $$\int_{-3}^{-2} \frac{1}{x} dx = \ln|x|\Big]_{-3}^{-2} = \ln 2 - \ln 3 = -0.4054...$$

THEOREM: If F is an antiderivative of f on an (open) interval I, then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

NOTATION: To denote the set of all antiderivatives of f on an (open) interval I we use the indefinite integral notation:

$$\int f(x)dx = F(x) + C$$

Table of Indefinite Integrals

$\int cf(x)dx = c\int f(x)dx$	$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{x} dx = \ln\|x\| + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \csc x \cot x dx = -\csc x + C$
$\int \frac{1}{1+x^2} dx = \arctan x + C$	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

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EXAMPLES:

1. $\displaystyle\int x^2 dx = [PR\ with\ n=2] = \frac{x^{2+1}}{2+1} + C = \frac{x^3}{3} + C$

2. $\displaystyle\int x^{-8} dx = [PR\ with\ n=-8] = \frac{x^{-8+1}}{-8+1} + C = \frac{x^{-7}}{-7} + C = -\frac{1}{7}x^{-7} + C$

3. $\displaystyle\int \sqrt{x}dx = \int x^{1/2}dx = [PR\ with\ n=1/2] = \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$

4. $\displaystyle\int \frac{1}{\sqrt[5]{x}}dx = \int \frac{1}{x^{1/5}}dx = \int x^{-1/5}dx = [PR\ with\ n=-1/5] = \frac{x^{-1/5+1}}{-1/5+1} + C = \frac{5}{4}x^{4/5} + C$

5. $\displaystyle\int x\sqrt[3]{x}dx = \int x^1 \cdot x^{1/3}dx = \int \underbrace{x^{1+1/3}}_{x^{4/3}}dx = [PR\ with\ n=4/3] = \frac{x^{4/3+1}}{4/3+1} + C = \frac{3}{7}x^{7/3} + C$

6. $\displaystyle\int \frac{x^4}{\sqrt[5]{x^3}}dx = \int \frac{x^4}{x^{3/5}}dx = \int \underbrace{x^{4-3/5}}_{x^{17/5}}dx = [PR\ with\ n=17/5] = \frac{x^{17/5+1}}{17/5+1} + C = \frac{5}{22}x^{22/5} + C$

7. $\displaystyle\int \frac{x\sqrt[3]{x}}{\sqrt{x}}dx$

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$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{x}dx = \ln|x| + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
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$\int \frac{1}{1+x^2}dx = \arctan x + C$	$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$

7. $$\int \frac{x\sqrt[3]{x}}{\sqrt{x}}dx = \int \frac{x^1 \cdot x^{1/3}}{x^{1/2}}dx = \int \frac{x^{1+1/3}}{x^{1/2}}dx = \int \underbrace{x^{1+1/3-1/2}}_{x^{5/6}}dx = [PR \ with \ n = 5/6] = \frac{x^{5/6+1}}{5/6+1} + C = \frac{6}{11}x^{11/6} + C$$

8. $$\int x(1+x^2)dx = \int \left(x + x^3\right) dx = \int xdx + \int x^3 dx = \frac{x^2}{2} + \frac{x^4}{4} + C$$

9. $$\int_0^1 \sqrt{x}\left(\sqrt[3]{x\sqrt{x}} + \sqrt[4]{x}\right) dx = \int_0^1 x^{1/2}\left(\left(x^1 \cdot x^{1/2}\right)^{1/3} + x^{1/4}\right) dx = \int_0^1 x^{1/2}\left(\left(x^{1+1/2}\right)^{1/3} + x^{1/4}\right) dx$$

$$= \int_0^1 x^{1/2}\left(\left(x^{3/2}\right)^{1/3} + x^{1/4}\right) dx = \int_0^1 x^{1/2}\left(x^{3/2\cdot 1/3} + x^{1/4}\right) dx = \int_0^1 x^{1/2}\left(x^{1/2} + x^{1/4}\right) dx$$

$$= \int_0^1 \left(x^{1/2}\cdot x^{1/2} + x^{1/2}\cdot x^{1/4}\right) dx = \int_0^1 \left(x^{1/2+1/2} + x^{1/2+1/4}\right) dx = \int_0^1 \left(x + x^{3/4}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{x^{3/4+1}}{3/4+1}\right]_0^1 = \left[\frac{x^2}{2} + \frac{x^{7/4}}{7/4}\right]_0^1 = \left[\frac{1}{2}x^2 + \frac{4}{7}x^{7/4}\right]_0^1$$

$$= \left(\frac{1}{2}\cdot 1^2 + \frac{4}{7}\cdot 1^{7/4}\right) - \left(\frac{1}{2}\cdot 0^2 + \frac{4}{7}\cdot 0^{7/4}\right) = \frac{1}{2} + \frac{4}{7} = \frac{15}{14}$$

10. $$\int x(1+x^2)^2 dx$$

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$\int cf(x)dx = c\int f(x)dx$	$\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
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$\int \frac{1}{1+x^2}dx = \arctan x + C$	$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$

10. $$\int x(1+x^2)^2 dx = \int x\left(1+2x^2+x^4\right)dx = \int (x+2x^3+x^5)dx$$

$$= \int xdx + 2\int x^3dx + \int x^5dx = \frac{x^2}{2} + 2\frac{x^4}{4} + \frac{x^6}{6} + C = \frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + C$$

11. $$\int \left(\frac{11x^2 - 5x^{-1/2} + 4x - 2}{7\sqrt{x}}\right)dx = \int \left(\frac{11x^2 - 5x^{-1/2} + 4x - 2}{7x^{1/2}}\right)dx$$

$$= \int \left(\frac{11x^2}{7x^{1/2}} - \frac{5x^{-1/2}}{7x^{1/2}} + \frac{4x}{7x^{1/2}} - \frac{2}{7x^{1/2}}\right)dx = \int \left(\frac{11}{7}x^{2-1/2} - \frac{5}{7}x^{-1/2-1/2} + \frac{4}{7}x^{1-1/2} - \frac{2}{7}x^{-1/2}\right)dx$$

$$= \int \left(\frac{11}{7}x^{3/2} - \frac{5}{7}x^{-1} + \frac{4}{7}x^{1/2} - \frac{2}{7}x^{-1/2}\right)dx = \frac{11}{7}\int x^{3/2}dx - \frac{5}{7}\int x^{-1}dx + \frac{4}{7}\int x^{1/2}dx - \frac{2}{7}\int x^{-1/2}dx$$

$$= \frac{11}{7}\cdot\frac{x^{3/2+1}}{3/2+1} - \frac{5}{7}\ln\lvert x\rvert + \frac{4}{7}\cdot\frac{x^{1/2+1}}{1/2+1} - \frac{2}{7}\cdot\frac{x^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{11}{7}\cdot\frac{x^{5/2}}{5/2} - \frac{5}{7}\ln\lvert x\rvert + \frac{4}{7}\cdot\frac{x^{3/2}}{3/2} - \frac{2}{7}\cdot\frac{x^{1/2}}{1/2} + C$$

$$= \frac{11}{7}\cdot\frac{2}{5}x^{5/2} - \frac{5}{7}\ln\lvert x\rvert + \frac{4}{7}\cdot\frac{2}{3}x^{3/2} - \frac{2}{7}\cdot 2x^{1/2} + C$$

$$= \frac{22}{35}x^{5/2} - \frac{5}{7}\ln\lvert x\rvert + \frac{8}{21}x^{3/2} - \frac{4}{7}x^{1/2} + C$$

12. $$\int \left(\frac{4 - 5\sqrt{x} + 7x\sin x}{9x}\right)dx$$

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$\int \frac{1}{1+x^2}dx = \arctan x + C$	$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$

12. $$\int \left(\frac{4-5\sqrt{x}+7x\sin x}{9x}\right)dx = \int\left(\frac{4}{9x} - \frac{5x^{1/2}}{9x} + \frac{7x\sin x}{9x}\right)dx$$
$$= \int\left(\frac{4}{9}\cdot\frac{1}{x} - \frac{5}{9}x^{-1/2} + \frac{7}{9}\sin x\right)dx$$
$$= \frac{4}{9}\int\frac{1}{x}dx - \frac{5}{9}\int x^{-1/2}dx + \frac{7}{9}\int \sin x dx$$
$$= \frac{4}{9}\ln|x| - \frac{5}{9}\cdot\frac{x^{-1/2+1}}{-1/2+1} - \frac{7}{9}\cos x + C$$
$$= \frac{4}{9}\ln|x| - \frac{10}{9}x^{1/2} - \frac{7}{9}\cos x + C$$

13. $$\int_0^1 \frac{x^3+x+1}{x^2+1}dx$$

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13. $$\int_0^1 \frac{x^3+x+1}{x^2+1}dx = \int_0^1 \left(\frac{x^3+x}{x^2+1} + \frac{1}{x^2+1}\right)dx = \int_0^1 \left(\frac{x(x^2+1)}{x^2+1} + \frac{1}{x^2+1}\right)dx = \int_0^1 \left(x + \frac{1}{x^2+1}\right)dx$$

$$= \left[\frac{x^2}{2} + \arctan x\right]_0^1 = \left(\frac{1^2}{2} + \arctan 1\right) - \left(\frac{0^2}{2} + \arctan 0\right) = \frac{1}{2} + \arctan 1 = \frac{1}{2} + \frac{\pi}{4}$$

14. $$\int 2\sec x(3\sec x - 5\tan x)dx = \int (6\sec^2 x - 10\sec x\tan x)dx = 6\int \sec^2 x dx - 10\int \sec x \tan x dx$$

$$= 6\tan x - 10\sec x + C$$

15. $$\int \frac{\sin 2x}{\sin x}dx$$

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15. $\int \frac{\sin 2x}{\sin x}dx = \int \frac{2\sin x\cos x}{\sin x}dx = \int 2\cos x dx = 2\int \cos x dx = 2\sin x + C$

16_1. $\int_0^{\pi/3} \frac{\cos x + \cos x\tan^2 x}{2\sec^2 x}dx = \int_0^{\pi/3} \frac{\cos x(1+\tan^2 x)}{2\sec^2 x}dx = \int_0^{\pi/3} \frac{\cos x\sec^2 x}{2\sec^2 x}dx = \int_0^{\pi/3} \frac{\cos x}{2}dx$

$$= \frac{1}{2}\int_0^{\pi/3} \cos x dx = \frac{1}{2}\sin x\Big]_0^{\pi/3} = \frac{1}{2}\left(\sin\left(\frac{\pi}{3}\right) - \sin 0\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0\right) = \frac{\sqrt{3}}{4}$$

16_2. $\int_0^{\pi/3} \frac{\cos x + \cos x\tan^2 x}{2\sec^2 x}dx = \int_0^{\pi/3} \frac{\cos x(1+\tan^2 x)}{2\sec^2 x}dx = \int_0^{\pi/3} \frac{\cos x\left(1+\frac{\sin^2 x}{\cos^2 x}\right)}{2\cdot\frac{1}{\cos^2 x}}dx$

$$= \int_0^{\pi/3} \frac{\cos x\left(\frac{\cos^2 x}{\cos^2 x}+\frac{\sin^2 x}{\cos^2 x}\right)}{2\cdot\frac{1}{\cos^2 x}}dx = \int_0^{\pi/3} \frac{\cos x\left(\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\right)}{2\cdot\frac{1}{\cos^2 x}}dx = \int_0^{\pi/3} \frac{\cos x\cdot\frac{1}{\cos^2 x}}{2\cdot\frac{1}{\cos^2 x}}dx = \int_0^{\pi/3} \frac{\cos x}{2}dx$$

$$= \frac{1}{2}\int_0^{\pi/3} \cos x dx = \frac{1}{2}\sin x\Big]_0^{\pi/3} = \frac{1}{2}\left(\sin\left(\frac{\pi}{3}\right) - \sin 0\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 0\right) = \frac{\sqrt{3}}{4}$$

17. $\int \left(x^2 + 2^x\right)dx = \int x^2 dx + \int 2^x dx = \frac{x^3}{3} + \frac{2^x}{\ln 2} + C$

Applications

THEOREM (Net Change Theorem): The integral of a rate of change is the net change:

$$\boxed{\int_a^b F'(x)dx = F(b) - F(a)}$$

EXAMPLE: We have

$$\int_a^b \left(\frac{x^3}{3}\right)' dx = \int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$$

APPLICATIONS:

(a) If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\boxed{\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)}$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2. Similarly,

$$\boxed{\int_{t_1}^{t_2} |v(t)|dt = \text{ total distance traveled}}$$

(b) The acceleration of the object is $a(t) = v'(t)$, so

$$\boxed{\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1)}$$

is the change in velocity from time t_1 to time t_2.

EXAMPLE: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

(b) Find the distance traveled during this time period.

EXAMPLE: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

Solution:

(a) The displacement is

$$s(4) - s(1) = \int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}$$

This means that the particle's position at time $t = 4$ is 4.5 m to the left of its position at the start of the time period.

(b) Note that

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$$

and so $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$ on $[3, 4]$. Thus the distance traveled is

$$\begin{aligned}\int_1^4 |v(t)|dt = \int_1^3 |v(t)|dt + \int_3^4 |v(t)|dt &= \int_1^3 [-v(t)]dt + \int_3^4 v(t)dt \\ &= \int_1^3 (-t^2 + t + 6)dt + \int_3^4 (t^2 - t - 6)dt \\ &= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \frac{61}{6} \approx 10.17 \text{ m}\end{aligned}$$