The Definite Integral

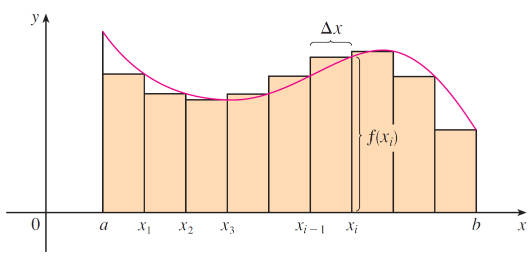
DEFINITION: The **area** A of the region S that lies under the graph of the continuous nonnegative function f is the limit of the sum of the areas of approximating rectangles:

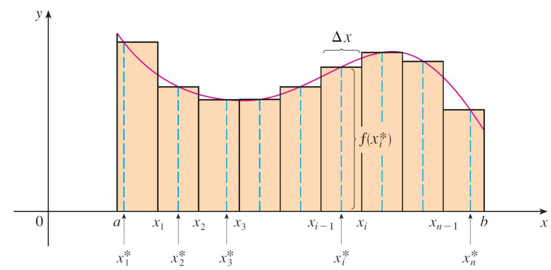
$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad (1)$$

REMARK 1: It can be proved that the above limit always exists, since we are assuming that f is continuous.

REMARK 2: It can also be shown that we get the same value if we use left endpoints or midpoints. Moreover, instead of using left endpoints, right endpoints or midpoints, we could take the height of the ith rectangle to be the value of f at *any* number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers $x_1^*, x_2^*, \ldots, x_n^*$ the **sample points**. So a more general expression for the area of S is

$$A = \lim_{n\to\infty}[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad (2)$$



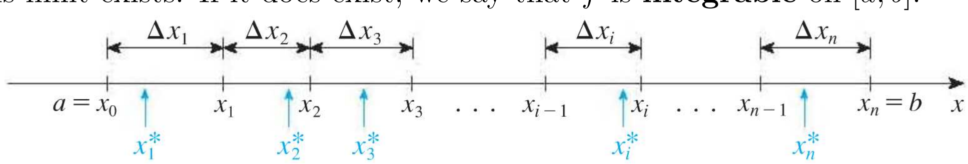


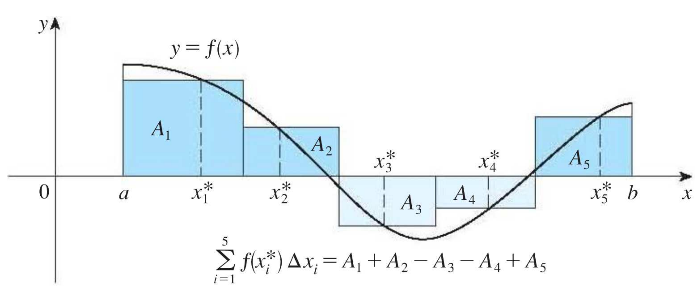
In this section we consider limits similar to (2) but in which f need not to be positive or continuous and the subintervals don't necessarily have the same length.

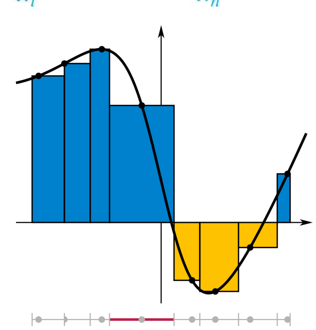
DEFINITION OF A DEFINITE INTEGRAL: If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is a number

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \underbrace{\sum_{i=1}^{n} f(x_i^*)\Delta x_i}_{Riemann\ sum}$$

provided that this limit exists. If it does exist, we say that f is **integrable** on $[a, b]$.







THEOREM: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int\limits_a^b f(x)dx$ exists.

THEOREM: If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

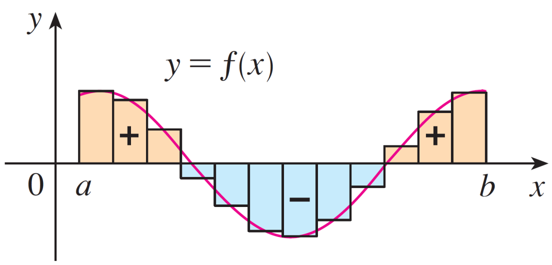
where

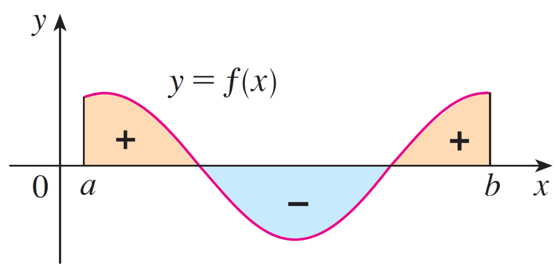
$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x$$

REMARK: A definite integral can be interpreted as a **net area,** that is, a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.





EXAMPLE: Evaluate $\int_0^4 (x^3 - 2x)dx$.

EXAMPLE: Evaluate $\displaystyle\int_0^4 (x^3 - 2x)dx$.

Solution: We partition $[0, 4]$ into n subintervals. We have

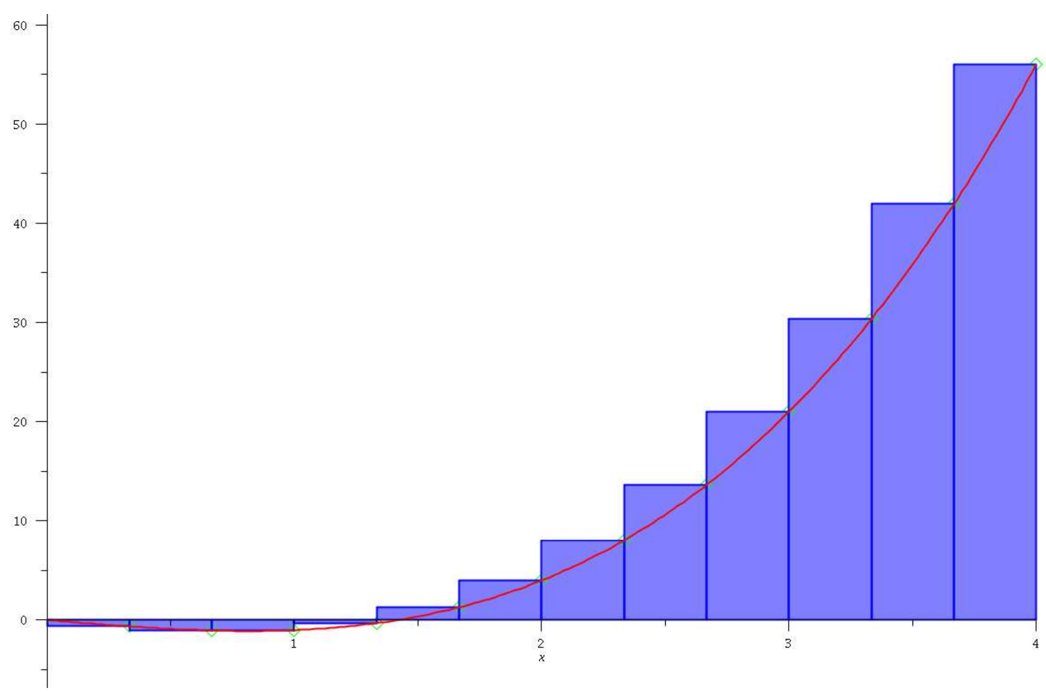
$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

Thus

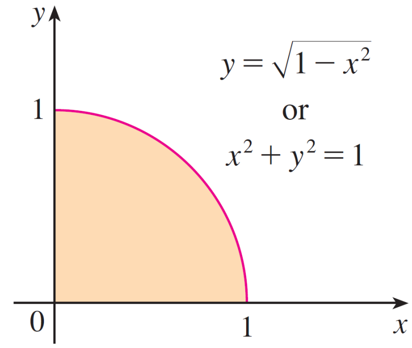
$$x_1 = \frac{4}{n},\ x_2 = \frac{8}{n}, \ldots, x_i = \frac{4i}{n}, \ldots, x_n = \frac{4n}{n} = 4$$

Therefore

$$\begin{aligned}
\int_0^4 (x^3 - 2x)dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(\frac{4i}{n}\right)\frac{4}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{4i}{n}\right)^3 - 2\frac{4i}{n}\right]\frac{4}{n} = \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^n \left[\left(\frac{4i}{n}\right)^3 - 2\frac{4i}{n}\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \sum_{i=1}^n \left[\frac{64i^3}{n^3} - \frac{8i}{n}\right] = \lim_{n\to\infty} \frac{4}{n} \left[\sum_{i=1}^n \frac{64i^3}{n^3} - \sum_{i=1}^n \frac{8i}{n}\right] \\
&= \lim_{n\to\infty} \frac{4}{n} \left[\frac{64}{n^3} \sum_{i=1}^n i^3 - \frac{8}{n} \sum_{i=1}^n i\right] = \lim_{n\to\infty} \frac{4}{n} \left[\frac{64}{n^3}\frac{n^2(n+1)^2}{4} - \frac{8}{n}\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[\frac{64(n+1)^2}{n^2} - \frac{16(n+1)}{n}\right] = \lim_{n\to\infty} \left[64\left(1+\frac{1}{n}\right)^2 - 16\left(1+\frac{1}{n}\right)\right] \\
&= 64 - 16 = 48
\end{aligned}$$



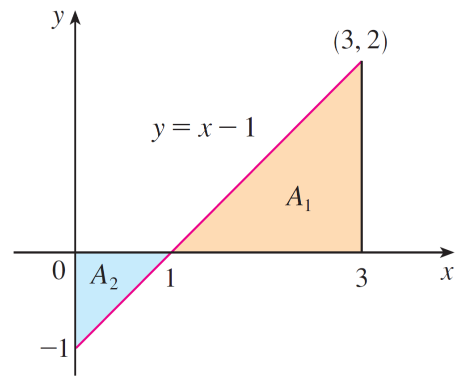
EXAMPLE: Evaluate $\int\limits_0^1 \sqrt{1-x^2}dx$ by interpreting it in terms of areas.



Solution: We have

$$\int\limits_0^1 \sqrt{1-x^2}dx = \frac{1}{4}\pi \cdot 1^2 = \frac{\pi}{4}$$

EXAMPLE: Evaluate $\int\limits_0^3 (x-1)dx$ by interpreting it in terms of areas.



Solution: We have

$$\int\limits_0^3 (x-1)dx = A_1 - A_2 = \frac{1}{2}(2\cdot 2) - \frac{1}{2}(1\cdot 1) = 1.5$$

EXAMPLE: Evaluate $\int_1^5 3dx$.

EXAMPLE: Evaluate $\int_0^4 xdx$.

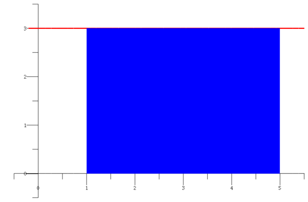
EXAMPLE: Evaluate $\int_2^7 xdx$.

EXAMPLE: Evaluate $\int_1^4 (2x+1)dx$.

EXAMPLE: Evaluate $\int_{-1}^4 (2x+1)dx$.

EXAMPLE: Evaluate $\int_{-2}^2 \sqrt{4-x^2}dx$.

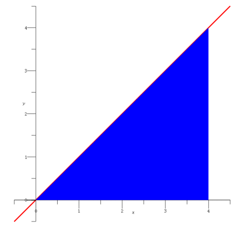
EXAMPLE: Evaluate $\int_1^5 3dx$.



Solution: We have

$$\int_1^5 3dx = (5-1)\cdot 3 = 12$$

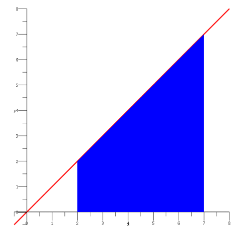
EXAMPLE: Evaluate $\int_0^4 xdx$.



Solution: We have

$$\int_0^4 xdx = \frac{4\cdot 4}{2} = 8$$

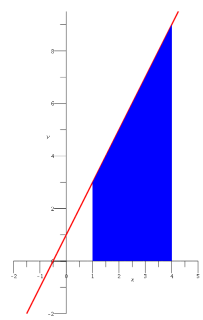
EXAMPLE: Evaluate $\int_2^7 xdx$.



Solution: We have

$$\int_2^7 xdx = \frac{(2+7)}{2}\cdot(7-2) = \frac{9}{2}\cdot 5 = \frac{45}{2}$$

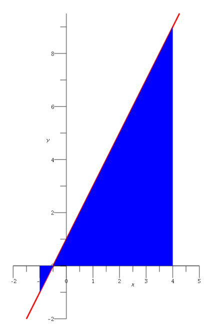
EXAMPLE: Evaluate $\int_1^4 (2x+1)dx$.



Solution: We have

$$\int_1^4 (2x+1)dx = \frac{(3+9)}{2}\cdot(4-1) = \frac{12}{2}\cdot 3 = 18$$

EXAMPLE: Evaluate $\int_{-1}^4 (2x+1)dx$.



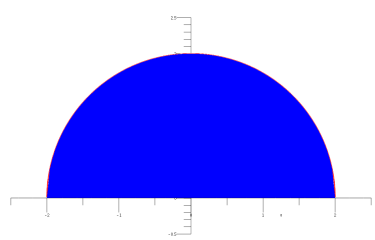
Solution 1: We have

$$\int_{-1}^4 (2x+1)dx = \frac{(-1+9)}{2}\cdot(4-(-1)) = \frac{8}{2}\cdot 5 = 20$$

Solution 2: We have

$$\int_{-1}^4 (2x+1)dx = -\frac{\frac{1}{2}\cdot 1}{2} + \frac{\frac{9}{2}\cdot 9}{2} = -\frac{1}{4} + \frac{81}{4} = \frac{-1+81}{4} = \frac{80}{4} = 20$$

EXAMPLE: Evaluate $\int_{-2}^2 \sqrt{4-x^2}dx$.



Solution: We have

$$\int_{-2}^2 \sqrt{4-x^2}dx = \frac{1}{2}\pi\cdot 2^2 = 2\pi$$

The Midpoint Rule

MIDPOINT RULE:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(\overline{x}_i)\Delta x = \Delta x[f(\overline{x}_1) + \ldots + f(\overline{x}_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

EXAMPLE: Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}dx$.

Solution: The endpoints of the five subintervals are

$$1, \quad 1.2, \quad 1.4, \quad 1.6, \quad 1.8, \quad \text{and} \quad 2.0$$

so the midpoints are

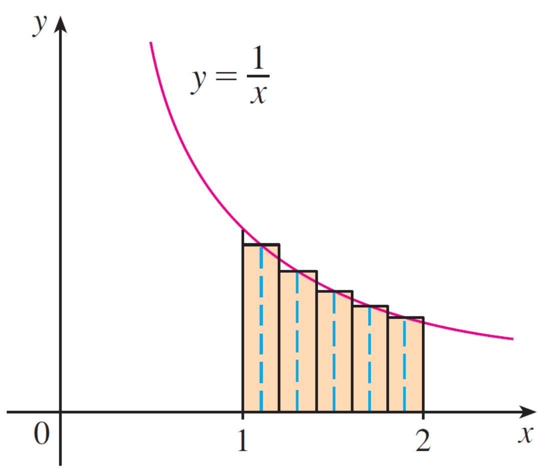
$$1.1, \quad 1.3, \quad 1.5, \quad 1.7, \quad \text{and} \quad 1.9$$

The width of the subintervals is $\Delta x = \frac{2-1}{5} = \frac{1}{5}$, so the Midpoint Rule gives

$$\begin{aligned}\int_1^2 \frac{1}{x}dx &\approx \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \\ &= \frac{1}{5}\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right] \approx 0.691908\end{aligned}$$

Note that

$$L_5 \approx 0.745635, \qquad R_5 \approx 0.645635, \qquad \int_1^2 \frac{1}{x}dx = \ln 2 \approx 0.693147$$

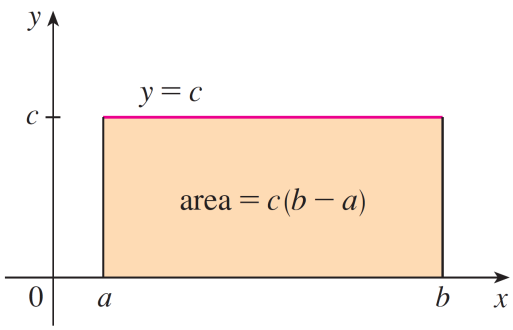


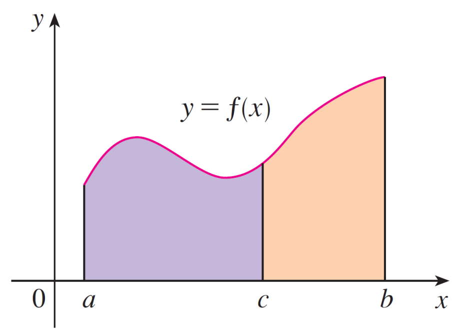
Properties of the Definite Integral

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b c\,dx = c(b-a)$





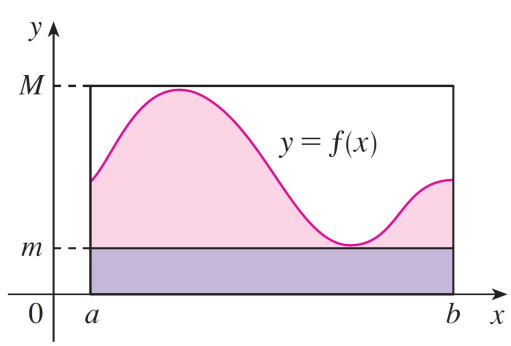
4. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

5. $\int_a^b [c_1 f(x) \pm c_2 g(x)]dx = c_1 \int_a^b f(x)dx \pm c_2 \int_a^b g(x)dx$

6. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

7. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

8. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int\limits_a^b f(x)dx \leq M(b-a)$.



EXAMPLE: Use Property 8 to estimate $\int\limits_1^4 \sqrt{x}dx$.

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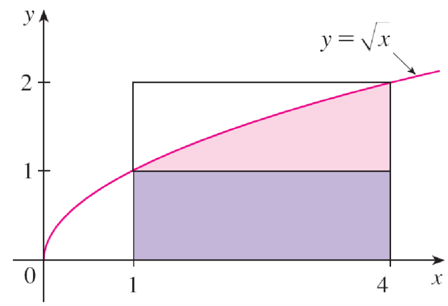
Solution: Since $f(x) = \sqrt{x}$ is an increasing function on $[1, 4]$, its absolute minimum value on $[1, 4]$ is

$$m = f(1) = \sqrt{1} = 1$$

and its absolute maximum value on $[1, 4]$ is

$$M = f(4) = \sqrt{4} = 2$$



Thus, by Property 8,

$$1(4-1) \le \int\limits_1^4 \sqrt{x}dx \le 2(4-1)$$

or

$$3 \le \int\limits_1^4 \sqrt{x}dx \le 6$$

EXAMPLE: Use Property 8 to estimate $\int\limits_0^1 e^{-x^2}dx$.

Solution: Since $f(x) = e^{-x^2}$ is a decreasing function on $[0, 1]$, its absolute maximum value on $[0, 1]$ is

$$M = f(0) = e^{-0^2} = 1$$

and its absolute minimum value on $[0, 1]$ is

$$m = f(1) = e^{-1^2} = e^{-1}$$

Thus, by Property 8,

$$e^{-1}(1-0) \le \int\limits_0^1 e^{-x^2}dx \le 1(1-0)$$

or

$$e^{-1} \le \int\limits_0^1 e^{-x^2}dx \le 1$$

Since $e^{-1} \approx 0.3679$, we can write

$$0.367 \le \int\limits_0^1 e^{-x^2}dx \le 1$$