Antiderivatives

DEFINITION: A function F is called an **antiderivative** of f on an (open) interval I if

$$F'(x) = f(x) \text{ for all } x \text{ in } I$$

EXAMPLES:

1. If $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$, since

$$F'(x) = \left(\frac{1}{3}x^3\right)' = \frac{1}{3}\left(x^3\right)' = \frac{1}{3} \cdot 3x^2 = x^2$$

2. If $f(x) = x$, then $F(x) = \frac{1}{2}x^2$, since

$$F'(x) = \left(\frac{1}{2}x^2\right)' = \frac{1}{2}\left(x^2\right)' = \frac{1}{2} \cdot 2x = x$$

3. If $f(x) = x^4$, then $F(x) = \frac{1}{5}x^5$, since

$$F'(x) = \left(\frac{1}{5}x^5\right)' = \frac{1}{5}\left(x^5\right)' = \frac{1}{5} \cdot 5x^4 = x^4$$

IN GENERAL:

$$\boxed{\text{If } f(x) = x^n, \text{ then } F(x) = \frac{x^{n+1}}{n+1}, \quad n \neq -1}$$

4. If $f(x) = \frac{1}{x}$, then $F(x) = \ln|x|$, since

$$F'(x) = (\ln|x|)' = \begin{cases} (\ln x)' = \dfrac{1}{x} & \text{if } x > 0 \\ \\ (\ln(-x))' = \dfrac{1}{-x} \cdot (-x)' = \dfrac{1}{-x} \cdot (-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

5. If $f(x) = \sin x$, then $F(x) = -\cos x$, since $F'(x) = (-\cos x)' = -(-\sin x) = \sin x$

6. If $f(x) = \cos x$, then $F(x) = \sin x$, since $F'(x) = (\sin x)' = \cos x$

7. If $f(x) = e^x$, then $F(x) = e^x$, since $F'(x) = (e^x)' = e^x$

8. If $f(x) = e^{x^2}$, then $F(x) =???$

9. If $f(x) = 1$, then $F(x) = x$, since $F'(x) = x' = 1$

10. If $f(x) = 0$, then $F(x) = 1$, since $F'(x) = 1' = 0$

REMARK: Obviously, in the last example any constant C can be used instead of 1, since $C' = 0$. This observation suggests that the previous examples can also be generalized. In fact, for instance, if $f(x) = x^2$ and $F(x) = \dfrac{1}{3}x^3$, then $F(x) + C$ is again an antiderivative of $f(x)$, since

$$(F(x) + C)' = \left(\frac{1}{3}x^3 + C\right)' = \left(\frac{1}{3}x^3\right)' + C' = x^2 + 0 = x^2$$

THEOREM: If F is an antiderivative of f on an (open) interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Proof: Let F and G be two functions such that $F' = f$ and $G' = f$ for all x from I. So, $F'(x) = G'(x)$ for all x from I, therefore by the Corollary from Section 4.2 $F(x) = G(x) + C$.

COROLLARY: If $F'(x) = G'(x)$ for all x in an interval (a, b), then $F - G$ is constant on (a, b); that is, $F(x) = G(x) + C$ where C is a constant.

Proof: Let $H(x) = F(x) - G(x)$. Then

$$H'(x) = F'(x) - G'(x) = 0$$

for all x in (a, b). Thus, by the Theorem below, H is constant; that is, $F - G$ is constant.

THEOREM: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof: Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. Therefore we can apply the Mean Value Theorem to f on the interval $[x_1, x_2]$ by which there exists a number c in (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $f'(x) = 0$ for all x, we have

$$f(x_2) - f(x_1) = 0 \cdot (x_2 - x_1) = 0 \quad \Longrightarrow \quad f(x_2) = f(x_1)$$

Therefore, f has the same value at *any* two numbers x_1 and x_2 in (a, b). This means that f is constant on (a, b).

THE MEAN VALUE THEOREM: Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or, equivalently, $f(b) - f(a) = f'(c)(b - a)$.

TABLE OF ANTIDIFFERENTIATION FORMULAS

Function	Particular Antiderivative	Function	Particular Antiderivative
$cf(x)$	$cF(x)$	$\sec^2 x$	$\tan x$
$f(x) \pm g(x)$	$F(x) \pm G(x)$	$\csc^2 x$	$-\cot x$
x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$1/x$	$\ln \lvert x \rvert$	$\csc x \cot x$	$-\csc x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
$\sin x$	$-\cos x$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\cos x$	$\sin x$		

EXAMPLE: Find all functions g such that $g'(x) = 2x^2 + 1$.

Solution: Using the Table and Theorem above, we have $g(x) = 2 \cdot \dfrac{x^3}{3} + x + C = \dfrac{2}{3}x^3 + x + C$.

EXAMPLE: Find all functions g such that $g'(x) = 5\cos x - \dfrac{8x^2 + 7\sqrt{x}}{9x^4}$.

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EXAMPLE: Find all functions g such that $g'(x) = 5\cos x - \dfrac{8x^2 + 7\sqrt{x}}{9x^4}$.

Solution: We have

$$g'(x) = 5\cos x - \frac{8x^2 + 7\sqrt{x}}{9x^4} = 5\cos x - \left(\frac{8x^2}{9x^4} + \frac{7x^{1/2}}{9x^4}\right) = 5\cos x - \frac{8}{9}x^{-2} - \frac{7}{9}x^{-7/2}$$

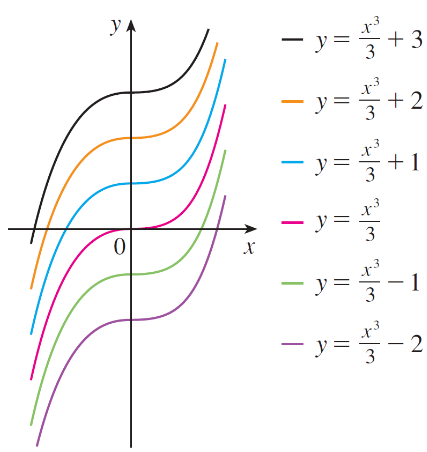
therefore by the Table and Theorem above we have

$$g(x) = 5\sin x - \frac{8}{9} \cdot \frac{x^{-2+1}}{-2+1} - \frac{7}{9} \cdot \frac{x^{-7/2+1}}{-7/2+1} + C = 5\sin x - \frac{8}{9} \cdot \frac{x^{-1}}{-1} - \frac{7}{9} \cdot \frac{x^{-5/2}}{-5/2} + C$$

$$= 5\sin x + \frac{8}{9}x^{-1} + \frac{7}{9} \cdot \frac{2}{5} \cdot x^{-5/2} + C = 5\sin x + \frac{8}{9}x^{-1} + \frac{14}{45}x^{-5/2} + C$$

EXAMPLE: Find f if

$$f'(x) = x^2 \quad \text{and} \quad f(0) = 2$$



Solution: Since $f'(x) = x^2$, it follows that

$$f(x) = \frac{x^3}{3} + C$$

by the Table and Theorem above. To find C we use the fact that $f(0) = 2$:

$$\frac{0^3}{3} + C = 2 \quad \Longrightarrow \quad 0 + C = 2 \quad \Longrightarrow \quad C = 2$$

Thus $f(x) = \dfrac{x^3}{3} + 2$.

EXAMPLE: Find f if

$$f'(x) = 5e^x - 4(1+x^2)^{-1} \quad \text{and} \quad f(0) = 1$$

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Solution: By the Table and Theorem above the general antiderivative of

$$f'(x) = 5e^x - 4(1+x^2)^{-1}$$

is

$$f(x) = 5e^x - 4\tan^{-1} x + C$$

To find C we use the fact that $f(0) = 1$:

$$5e^0 - 4\tan^{-1} 0 + C = 1 \quad \Longrightarrow \quad 5\cdot 1 - 4\cdot 0 + C = 1 \quad \Longrightarrow \quad 5 + C = 1 \quad \Longrightarrow \quad C = -4$$

Thus

$$f(x) = 5e^x - 4\tan^{-1} x - 4$$

EXAMPLE: Find f if

$$f''(x) = x,\ f'(2) = 5,\ \text{and } f(0) = 2$$

Solution: By the Table and Theorem above we have

$$f''(x) = x \quad \Longrightarrow \quad f'(x) = \frac{1}{2}x^2 + C_1$$

From this and $f'(2) = 5$ it follows that

$$\frac{1}{2}\cdot 2^2 + C_1 = 5 \quad \Longrightarrow \quad 2 + C_1 = 5 \quad \Longrightarrow \quad C_1 = 3 \quad \Longrightarrow \quad f'(x) = \frac{1}{2}x^2 + 3$$

If we apply the Table and Theorem above again, we get

$$f'(x) = \frac{1}{2}x^2 + 3 \quad \Longrightarrow \quad f(x) = \frac{1}{6}x^3 + 3x + C_2$$

From this and $f(0) = 2$ it follows that

$$\frac{1}{6}\cdot 0^3 + 3\cdot 0 + C_2 = 2 \quad \Longrightarrow \quad C_2 = 2$$

Thus

$$f(x) = \frac{1}{6}x^3 + 3x + 2$$

EXAMPLE: Find f if

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$$f''(x) = 12x^2 + 6x - 4, \ f(0) = 4, \ \text{and} \ f(1) = 1$$

Solution: The general antiderivative of $f''(x) = 12x^2 + 6x - 4$ is

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C = 4x^3 + 3x^2 - 4x + C$$

Using the antidifferentiation rules once more, we find that

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

To determine C and D we use the given conditions that $f(0) = 4$ and $f(1) = 1$. Since $f(0) = 0 + D = 4$, we have $D = 4$. Since

$$f(1) = 1 + 1 - 2 + C + 4 = 1$$

we have $C = -3$. Therefore the required function is

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

Rectilinear Motion

EXAMPLE: A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

EXAMPLE: A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

Solution: Since $v'(t) = a(t) = 6t + 4$, antidifferentiation gives

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$$

Note that $v(0) = C$. But we are given that $v(0) = -6$, so $C = -6$ and

$$v(t) = 3t^2 + 4t - 6$$

Since $v(t) = s'(t)$, s is the antiderivative of v:

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D$$

This gives $s(0) = D$. We are given that $s(0) = 9$, so $D = 9$ and the required position function is

$$s(t) = t^3 + 2t^2 - 6t + 9$$

EXAMPLE: A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? What is its maximum height? When does it hit the ground?

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Solution: An object near the surface of the Earth is subject to a gravitational force that produces a downward acceleration denoted by g. For motion close to the ground we may assume that g is constant, its value being about 9.8 m/s^2 (or 32 ft/s^2).

The motion is vertical and we choose the positive direction to be upward. At time t the distance above the ground is $s(t)$ and the velocity $v(t)$ is decreasing. Therefore, the acceleration must be negative and we have

$$a(t) = \frac{dv}{dt} = -32$$

Taking antiderivatives, we have

$$v(t) = -32t + C$$

To determine C we use the information that $v(0) = 48$. This gives $48 = 0 + C$, so

$$v(t) = -32t + 48$$

The maximum height is reached when $v(t) = 0$, that is, after 1.5 s. As $s'(t) = v(t)$, we antidifferentiate again and obtain

$$s(t) = -16t^2 + 48t + D$$

Using the fact that $s(0) = 432$, we have $432 = 0 + D$ and so

$$s(t) = -16t^2 + 48t + 432$$

Therefore the maximum height is

$$s(1.5) = -16(1.5)^2 + 48 \cdot 1.5 + 432 = 468 \text{ ft}$$

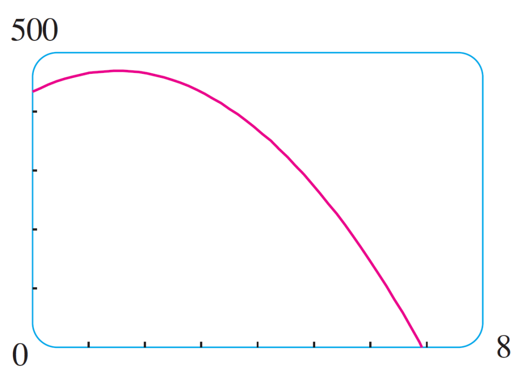
The expression for $s(t)$ is valid until the ball hits the ground. This happens when $s(t) = 0$, that is, when

$$-16t^2 + 48t + 432 = 0$$

or, equivalently,

$$t^2 - 3t - 27 = 0$$

Using the quadratic formula to solve this equation, we get $t = \dfrac{3 \pm 3\sqrt{13}}{2}$. We reject the solution with the minus sign since it gives a negative value for t. Therefore, the ball hits the ground after $3(1 + \sqrt{13})/2 \approx 6.9$ s.



Appendix

1. If $f(x) = \dfrac{1}{1+x}$, then

$$f'(x) = ((1+x)^{-1})' = (-1)(1+x)^{-2} \cdot (1+x)' = (-1)(1+x)^{-2} \cdot 1 = \boxed{-\frac{1}{(1+x)^2}}$$

and $\boxed{F(x) = \ln|1+x|}$.

2. If $f(x) = \dfrac{1}{1+x^2}$, then

$$f'(x) = ((1+x^2)^{-1})' = (-1)(1+x^2)^{-2} \cdot (1+x^2)' = (-1)(1+x^2)^{-2} \cdot 2x = \boxed{-\frac{2x}{(1+x^2)^2}}$$

and $\boxed{F(x) = \tan^{-1} x}$.

3. If $f(x) = \dfrac{1}{1+x^3}$, then

$$f'(x) = ((1+x^3)^{-1})' = (-1)(1+x^3)^{-2} \cdot (1+x^3)' = (-1)(1+x^3)^{-2} \cdot 3x^2 = \boxed{-\frac{3x^2}{(1+x^3)^2}}$$

and

$$\boxed{F(x) = -\frac{1}{6}\ln\left(x^2 - x + 1\right) + \frac{1}{3}\ln(x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}$$

4. If $f(x) = \dfrac{1}{1+x^4}$, then

$$f'(x) = ((1+x^4)^{-1})' = (-1)(1+x^4)^{-2} \cdot (1+x^4)' = (-1)(1+x^4)^{-2} \cdot 4x^3 = \boxed{-\frac{4x^3}{(1+x^4)^2}}$$

and

$$\boxed{\begin{aligned} F(x) = \frac{1}{4\sqrt{2}}\Bigg(-\ln\left(x^2 - \sqrt{2}x + 1\right) + \ln\left(x^2 + \sqrt{2}x + 1\right) &-2\tan^{-1}\left(1 - \sqrt{2}x\right) \\ &+2\tan^{-1}\left(\sqrt{2}x + 1\right)\Bigg) \end{aligned}}$$

5. If $f(x) = \dfrac{1}{1+x^5}$, then

$$f'(x) = ((1+x^5)^{-1})' = (-1)(1+x^5)^{-2} \cdot (1+x^5)' = (-1)(1+x^5)^{-2} \cdot 5x^4 = \boxed{-\frac{5x^4}{(1+x^5)^2}}$$

and

$$\boxed{\begin{aligned} F(x) = \frac{1}{20}\Bigg(&\left(\sqrt{5}-1\right)\ln\left(x^2 + \frac{1}{2}\left(\sqrt{5}-1\right)x + 1\right) - \left(1+\sqrt{5}\right)\ln\left(x^2 - \frac{1}{2}\left(1+\sqrt{5}\right)x + 1\right) \\ &+4\ln(x+1) - 2\sqrt{10-2\sqrt{5}}\tan^{-1}\left(\frac{-4x+\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}\right) + 2\sqrt{2\left(5+\sqrt{5}\right)}\tan^{-1}\left(\frac{4x+\sqrt{5}-1}{\sqrt{2\left(5+\sqrt{5}\right)}}\right)\Bigg) \end{aligned}}$$