Optimization Problems

EXAMPLE 1: A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution: Note that the area of the field depends on its dimensions:

To solve the problem, we first draw a picture that illustrates the general case:

The next step is to create a corresponding mathematical model:

Maximize: \( A = xy \)

Constraint: \( 2x + y = 2400 \)

To find the absolute maximum value of \( A = 2400x - 2x^2 \), we use

The Closed Interval Method: To find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b] \):

1. Find the values of \( f \) at the critical numbers of \( f \) in \((a, b)\).
2. Find the values of \( f \) at the endpoints of the interval.
3. The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest value of these values is the absolute minimum value.

We first show that \( 0 \leq x \leq 1200 \). Indeed,

\[
y \geq 0 \implies 2400 - 2x \geq 0 \implies 2400 \geq 2x \implies 1200 \geq x
\]

On the other hand, \( x \geq 0 \). Combining these two inequalities gives \( 0 \leq x \leq 1200 \). The derivative of \( A(x) \) is

\[
A'(x) = (2400x - 2x^2)' = 2400x' - 2(x^2)' = 2400 \cdot 1 - 2 \cdot 2x = 2400 - 4x
\]

so to find the critical numbers we solve the equation

\[
2400 - 4x = 0 \implies 2400 = 4x \implies x = \frac{2400}{4} = 600
\]

To find the maximum value of \( A(x) \) we evaluate it at the end points and critical number:

\[
A(0) = 0, \quad A(600) = 2400 \cdot 600 - 2 \cdot 600^2 = 720,000, \quad A(1200) = 0
\]

The Closed Interval Method gives the maximum value as \( A(600) = 720,000 \text{ ft}^2 \) and the dimensions are \( x = 600 \text{ ft}, y = 2400 - 2 \cdot 600 = 1200 \text{ ft} \).

EXAMPLE 2: We need to enclose a field with a rectangular fence. We have 500 ft of fencing material and a building is on one side of the field and so won’t need any fencing. Determine the dimensions of the field that will enclose the largest area.
EXAMPLE 2: We need to enclose a field with a rectangular fence. We have 500 ft of fencing material and a building is on one side of the field and so won’t need any fencing. Determine the dimensions of the field that will enclose the largest area.

Solution: We first draw a picture that illustrates the general case:

The next step is to create a corresponding mathematical model:

Maximize: \( A = xy \)

Constraint: \( x + 2y = 500 \)

We now solve the second equation for \( x \) and substitute the result into the first equation to express \( A \) as a function of one variable:

\[
x + 2y = 500 \implies x = 500 - 2y \implies A = xy = (500 - 2y)y = 500y - 2y^2
\]

To find the absolute maximum value of \( A = 500y - 2y^2 \), we use the Closed Interval Method. We first show that \( 0 \leq y \leq 250 \). Indeed,

\[
x \geq 0 \implies 500 - 2y \geq 0 \implies 500 \geq 2y \implies 250 \geq y
\]

On the other hand, \( y \geq 0 \). Combining these two inequalities gives \( 0 \leq y \leq 250 \). The derivative of \( A(y) \) is

\[
A'(y) = (500y - 2y^2)' = 500y' - 2(y^2)' = 500 \cdot 1 - 2 \cdot 2y = 500 - 4y
\]

so to find the critical numbers we solve the equation

\[
500 - 4y = 0 \implies 500 = 4y \implies y = \frac{500}{4} = 125
\]

To find the maximum value of \( A(y) \) we evaluate it at the end points and critical number:

\[
A(0) = 0, \quad A(125) = 500 \cdot 125 - 2 \cdot 125^2 = 31,250, \quad A(250) = 0
\]

The Closed Interval Method gives the maximum value as \( A(125) = 31,250 \) ft\(^2\) and the dimensions are \( y = 125 \) ft, \( x = 500 - 2 \cdot 125 = 250 \) ft.

EXAMPLE 3: We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $10/ft\(^2\) and the material used to build the sides cost $6/ft\(^2\). If the box must have a volume of 50 ft\(^3\) determine the dimensions that will minimize the cost to build the box.
EXAMPLE 3: We want to construct a box whose base length is 3 times the base width. The material used to build the top and bottom cost $10/ft^2$ and the material used to build the sides cost $6/ft^2$. If the box must have a volume of $50$ ft$^3$ determine the dimensions that will minimize the cost to build the box.

Solution: We first draw a picture:

The next step is to create a corresponding mathematical model:

Minimize: $C = 10(2lw + 2wh) + 6(2wh + 2·3w·h) = 10(2·3w·w) + 6(2wh + 2·3w·h) = 60w^2 + 48wh$

Constraint: $V = lwh = 3w·wh = 3w^2h = 50$

We now solve the second equation for $h$ and substitute the result into the first equation to express $C$ as a function of one variable:

$3w^2h = 50 \implies h = \frac{50}{3w^2} \implies C = 60w^2 + 48wh = 60w^2 + 48w·\frac{50}{3w^2} = 60w^2 + \frac{800}{w}$

Note that we can’t use the Closed Interval Method because the domain of $C(w)$ is $(0, \infty)$ which is not a finite interval. Instead, we will use

FIRST DERIVATIVE TEST FOR ABSOLUTE EXTREME VALUES: Suppose that $c$ is a critical number of a continuous function $f$ defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of $f$.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of $f$.

The derivative of $C(w)$ is

$$C'(w) = \left(60w^2 + \frac{800}{w}\right)' = 120w - \frac{800}{w^2} = \frac{120w^3 - 800}{w^2} \quad \text{(see Appendix for details)}$$

Since $w > 0$, the only critical number is $w = \sqrt[3]{\frac{800}{120}} = \sqrt[3]{\frac{20}{3}} \approx 1.8821$. It is easy to see that $C''(w) < 0$ for all $0 < w < \sqrt[3]{\frac{20}{3}}$ and $C''(w) > 0$ for all $w > \sqrt[3]{\frac{20}{3}}$. Therefore the minimum value of the cost must occur at $w = \sqrt[3]{\frac{20}{3}}$. The dimensions are

$$w = \sqrt[3]{\frac{20}{3}} \approx 1.8821 \text{ ft}, \quad l = 3w = 3\sqrt[3]{\frac{20}{3}} \approx 5.6463 \text{ ft}, \quad h = \frac{50}{3w^2} \approx 4.7050 \text{ ft}$$

and the minimum cost is $C\left(\sqrt[3]{\frac{20}{3}}\right) \approx 637.60$.

EXAMPLE 4: We want to construct a box with a square base and we only have 10 m$^2$ of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.
EXAMPLE 4: We want to construct a box with a square base and we only have 10 m\(^2\) of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

Solution: We first draw a picture:

![Diagram of a box with a square base and height labeled h, and length labeled w, with w = l = h]($l = w$

The next step is to create a corresponding mathematical model:

Maximize: \(V = lwh = w^2h\)

Constraint: \(2lw + 2wh + 2lh = 2w^2 + 2wh + 2wh = 2w^2 + 4wh = 10\)

We now solve the second equation for \(h\) and substitute the result into the first equation to express \(V\) as a function of one variable:

\[
2w^2 + 4wh = 10 \implies h = \frac{10 - 2w^2}{4w} = \frac{5 - w^2}{2w} \implies V = w^2h = w^2 \left(\frac{5 - w^2}{2w}\right) = \frac{1}{2}(5w - w^3)
\]

Since \(w > 0\), we can use only the First Derivative Test for Absolute Extreme Values. The derivative of \(V(w)\) is

\[
V'(w) = \left(\frac{1}{2}(5w - w^3)\right)' = \frac{1}{2}(5w - w^3)' = \frac{1}{2}(5 - 3w^2)
\]

Since \(w > 0\), the only critical number is \(w = \sqrt{\frac{5}{3}}\). It is easy to see that \(V'(w) > 0\) for all \(0 < w < \sqrt{\frac{5}{3}}\) and \(V'(w) < 0\) for all \(w > \sqrt{\frac{5}{3}}\). Therefore the maximum value of the volume must occur at \(w = \sqrt{\frac{5}{3}}\). Finally, the dimensions of the box are

\[
w = l = \sqrt{\frac{5}{3}} \approx 1.2910 \, \text{m}, \quad h = \frac{5 - w^2}{2w} \approx 1.2910 \, \text{m}
\]

which means the box with the maximum volume \(V = \left(\sqrt{\frac{5}{3}}\right)^3 \approx 2.1517 \, \text{m}^3\) is a cube.

EXAMPLE 5: A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.
EXAMPLE 5: A manufacturer needs to make a cylindrical can that will hold 1.5 liters of liquid. Determine the dimensions of the can that will minimize the amount of material used in its construction.

Solution: We first draw a picture:

![Diagram of a cylinder with dimensions labeled]

The next step is to create a corresponding mathematical model:

Minimize: \( A = 2\pi r^2 + 2\pi rh \)

Constraint: \( V = \pi r^2 h = 1500 \)

We now solve the second equation for \( h \) and substitute the result into the first equation to express \( A \) as a function of one variable:

\[
\pi r^2 h = 1500 \implies h = \frac{1500}{\pi r^2} \implies A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot \frac{1500}{\pi r^2} = 2\pi r^2 + \frac{3000}{r}
\]

To find the absolute minimum value of \( A = 2\pi r^2 + \frac{3000}{r} \), we use the First Derivative Test for Absolute Extreme Values. The derivative of \( A(r) \) is

\[
A'(r) = \left(2\pi r^2 + \frac{3000}{r}\right)' = 4\pi r - \frac{3000}{r^2} = \frac{4\pi r^3 - 3000}{r^2}
\]

(see Appendix for details)

Since \( r > 0 \), the only critical number is \( r = \sqrt[3]{\frac{3000}{4\pi}} = \sqrt[3]{\frac{750}{\pi}} \). It is easy to see that \( A'(r) < 0 \) for all \( 0 < r < \sqrt[3]{\frac{750}{\pi}} \) and \( A'(r) > 0 \) for all \( r > \sqrt[3]{\frac{750}{\pi}} \). Therefore the minimum value of the area must occur at \( r = \sqrt[3]{\frac{750}{\pi}} \approx 6.2035 \) cm and this value is

\[
A \left(\sqrt[3]{\frac{750}{\pi}}\right) \approx 725.3964 \text{ cm}^2
\]

Finally, the height of the can is

\[
h = \frac{1500}{\pi r^2} = \frac{1500}{\pi \left(\sqrt[3]{750}/\pi\right)^{2/3}} = 2r \approx 12.4070 \text{ cm}
\]
EXAMPLE 6: We have a piece of cardboard that is 14 in by 10 in and we’re going to cut out the corners as shown below and fold up the sides to form a box, also shown below. Determine the height of the box that will give a maximum volume.

Solution: We create a corresponding mathematical model:

Maximize: \( V = h(14 - 2h)(10 - 2h) = 140h - 48h^2 + 4h^3 \)

We first show that \( 0 \leq h \leq 5 \). Indeed,

width \geq 0 \implies 10 - 2h \geq 0 \implies 10 \geq 2h \implies 5 \geq h \)

On the other hand, \( h \geq 0 \). Combining these two inequalities gives \( 0 \leq h \leq 5 \). Therefore we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values to find the absolute maximum value of \( V = 140h - 48h^2 + 4h^3 \).

**Closed Interval Method:** The derivative of \( V(h) \) is

\[ V'(h) = (140h - 48h^2 + 4h^3)' = 140h' - 48(h^2)' + 4(h^3)' = 140 - 48\cdot2h + 4\cdot3h^2 = 140 - 96h + 12h^2 \]

so to find the critical numbers we solve the equation

\[ 140 - 96h + 12h^2 = 0 \implies h = \frac{-(-96) \pm \sqrt{(-96)^2 - 4\cdot12\cdot140}}{2\cdot12} = \frac{12 \pm \sqrt{39}}{3} \approx 1.9183, 6.0817 \]

Since \( 0 \leq h \leq 5 \), the only critical number that we must consider is \( h = \frac{12 \pm \sqrt{39}}{3} \approx 1.9183 \). To find the maximum value of \( V(h) \) we evaluate it at the end points and critical number:

\[ V(0) = 0, \quad V\left(\frac{12 - \sqrt{39}}{3}\right) \approx 120.1644, \quad V(5) = 0 \]

Therefore the maximum value of the volume must occur at \( h = \frac{12 - \sqrt{39}}{3} \approx 1.9183 \) in and this value is \( \approx 120.1644 \) in\(^3\).

**First Derivative Test for Absolute Extreme Values:** By the above, \( V'(h) = 140 - 96h + 12h^2 \) and the only critical number that we must consider is \( h = \frac{12 - \sqrt{39}}{3} \). It is easy to see that \( V''(h) > 0 \) for all \( h < \frac{12 - \sqrt{39}}{3} \) and \( V''(h) < 0 \) for all \( h > \frac{12 - \sqrt{39}}{3} \) from \([0,5]\). Therefore the maximum value of the volume must occur at \( h = \frac{12 - \sqrt{39}}{3} \approx 1.9183 \) in and this value is \( V\left(\frac{12 - \sqrt{39}}{3}\right) \approx 120.1644 \) in\(^3\).
EXAMPLE 7: A printer needs to make a poster that will have a total area of 200 in² and will have 1 in margins on the sides, a 2 in margin on the top and a 1.5 in margin on the bottom. What dimensions of the poster will give the largest printed area?

Solution: We first draw a picture. Then we create a corresponding mathematical model:

Maximize: \[ A = (w - 2)(h - 3.5) \]

Constraint: \( wh = 200 \)

We now solve the second equation for \( h \) and substitute the result into the first equation to express \( A \) as a function of one variable:

\[
wh = 200 \implies h = \frac{200}{w}
\]

so

\[
A = (w - 2)(h - 3.5) = (w - 2) \left( \frac{200}{w} - 3.5 \right) = 207 - 3.5w - \frac{400}{w}
\]

We show that \( 2 \leq w \leq \frac{200}{3.5} \). Indeed,

length \( \geq 0 \implies w - 2 \geq 0 \implies w \geq 2 \)

height \( \geq 0 \implies h - 3.5 \geq 0 \implies \frac{200}{w} - 3.5 \geq 0 \implies \frac{200}{w} \geq 3.5 \implies 200 \geq 3.5w \implies \frac{200}{3.5} \geq w \)

Combining these two inequalities gives \( 2 \leq w \leq \frac{200}{3.5} \). Therefore we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values to find the absolute maximum value of \( A = 207 - 3.5w - \frac{400}{w} \).

Closed Interval Method: The derivative of \( A(w) \) is

\[
A'(w) = \left( 207 - 3.5w - \frac{400}{w} \right)' = -3.5 + \frac{400}{w^2} = -3.5w^2 + \frac{400}{w^2}
\]

(see Appendix for details)

Since \( w \geq 2 \), the only critical number is \( w = \sqrt{400/3.5} \). To find the maximum value of \( A(w) \) we evaluate it at the end points and critical number:

\[
A(2) = 0, \quad A \left( \sqrt{400/3.5} \right) \approx 120.1644, \quad A \left( \frac{200}{3.5} \right) = 0
\]

Therefore the maximum value of the area must occur at \( w = \sqrt{400/3.5} \approx 10.6905 \) in and this value is \( \approx 132.1669 \) in². Finally, the height of the paper that gives the maximum printed area is

\[
h = \frac{200}{w} = \frac{200}{\sqrt{400/3.5}} = 10\sqrt{3.5} \approx 18.7083 \text{ in}
\]

First Derivative Test for Absolute Extreme Values: By the above, \( A'(w) = -3.5w^2 + \frac{400}{w^2} \) and the only critical number that we must consider is \( w = \sqrt{400/3.5} \). It is easy to see that \( A'(w) > 0 \) for all \( 2 \leq w < \sqrt{400/3.5} \) and \( A'(w) < 0 \) for all \( w > \sqrt{400/3.5} \). Therefore the maximum value of the area must occur at \( w = \sqrt{400/3.5} \approx 10.6905 \) in, \( h = 10\sqrt{3.5} \approx 18.7083 \) in and this value is \( \approx 132.1669 \) in².
EXAMPLE 8: A window is being built and the bottom is a rectangle and the top is a semicircle. If there is 12 m of framing materials what must the dimensions of the window be to let in the most light?

Solution: We first draw a picture. The next step is to create a corresponding mathematical model:

Maximize: \( A = 2hr + \frac{1}{2}\pi r^2 \)

Constraint: \( 2h + 2r + \pi r = 12 \)

We now solve the second equation for \( h \) and substitute the result into the first equation to express \( A \) as a function of one variable:

\[
2h + 2r + \pi r = 12 \quad \Rightarrow \quad h = 6 - r - \frac{1}{2}\pi r
\]

hence

\[
A = 2hr + \frac{1}{2}\pi r^2 = 2r \left( 6 - r - \frac{1}{2}\pi r \right) + \frac{1}{2}\pi r^2 = 12r - 2r^2 - \frac{1}{2}\pi r^2 = 12r - \left( 2 + \frac{1}{2}\pi \right) r^2
\]

We show that \( 0 \leq r \leq \frac{12}{2 + \pi} \). Indeed, since \( h \geq 0 \), we have

\[
6 - r - \frac{1}{2}\pi r \geq 0 \quad \Rightarrow \quad 6 \geq r + \frac{1}{2}\pi r \quad \Rightarrow \quad 6 \geq r \left( 1 + \frac{1}{2}\pi \right) \quad \Rightarrow \quad r \leq \frac{6}{1 + \frac{1}{2}\pi} = \frac{12}{2 + \pi}
\]

On the other hand, \( r \geq 0 \). Combining these two inequalities gives \( 0 \leq r \leq \frac{12}{2 + \pi} \). Therefore we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values to find the absolute maximum value of \( A = 12r - \left( 2 + \frac{1}{2}\pi \right) r^2 \).

Closed Interval Method: The derivative of \( A(r) \) is

\[
A'(r) = \left( 12r - \left( 2 + \frac{1}{2}\pi \right) r^2 \right)' = 12r' - \left( 2 + \frac{1}{2}\pi \right) (r^2)' = 12 - \left( 2 + \frac{1}{2}\pi \right) \cdot 2r = 12 - (4 + \pi)r
\]

To find the critical numbers we solve the equation

\[
12 - (4 + \pi)r = 0 \quad \Rightarrow \quad 12 = (4 + \pi)r \quad \Rightarrow \quad r = \frac{12}{4 + \pi}
\]

To find the maximum value of \( A(r) \) we evaluate it at the end points and critical number:

\[
A(0) = 0, \quad A \left( \frac{12}{4 + \pi} \right) \approx 10.0817, \quad A \left( \frac{12}{2 + \pi} \right) = \frac{72\pi}{(2 + \pi)^2} \approx 8.5563
\]

Therefore the maximum value of the area must occur at \( r = \frac{12}{4 + \pi} \approx 1.6803 \text{ m and this value is } A \left( \frac{12}{4 + \pi} \right) \approx 10.0817 \text{ m}^2 \). Finally, the height of the window that gives the maximum area is \( h = 2r = \frac{24}{4 + \pi} \approx 3.3606 \text{ m} \).

First Derivative Test for Absolute Extreme Values: By the above, \( A'(r) = 12 - (4 + \pi)r \) and the critical number is \( r = \frac{12}{4 + \pi} \). It is easy to see that \( A'(r) > 0 \) for all \( r < \frac{12}{4 + \pi} \) and \( A'(r) < 0 \) for all \( r > \frac{12}{4 + \pi} \). Therefore the maximum value of the area must occur at \( r = \frac{12}{4 + \pi} \approx 1.6803 \text{ m and this value is } A \left( \frac{12}{4 + \pi} \right) \approx 10.0817 \text{ m}^2 \); the height is \( h = 2r = \frac{24}{4 + \pi} \approx 3.3606 \text{ m} \).
EXAMPLE 9: Determine the area of the largest rectangle that can be inscribed in a circle of radius 4.

Solution: We first draw a picture:

The next step is to create a corresponding mathematical model:

Maximize: \( A = 2x \cdot 2y = 4xy \)

Constraint: \( x^2 + y^2 = 16 \)

We can solve the second equation for \( x \) and substitute the result into the first equation to express \( A \) as a function of one variable. However, this approach involves roots which makes the algebra a bit complicated.

Instead, we square both sides of \( A = 4xy \). Note that \( x \) and \( y \) are both nonnegative. Therefore values that maximize \( A = 4xy \) will also maximize \( A^2 = 16x^2y^2 \) and vice-versa. Putting \( B = A^2, u = x^2, v = y^2 \), we reformulate our problem in the following way:

Maximize: \( B = 16uv \)

Constraint: \( u + v = 16 \)

We now solve the second equation for \( u \) and substitute the result into the first equation to express \( B \) as a function of one variable:

\[ u + v = 16 \quad \Rightarrow \quad u = 16 - v \quad \Rightarrow \quad B = 16uv = 16(16 - v)v = 256v - 16v^2 \]

To find the absolute maximum value of \( B = 256v - 16v^2 \), we can use either the Closed Interval Method or the First Derivative Test for Absolute Extreme Values. Here we use the First Derivative Test for Absolute Extreme Values. The derivative of \( B(v) \) is \( B'(v) = 256 - 32v \), so to find the critical numbers we solve the equation

\[ 256 - 32v = 0 \quad \Rightarrow \quad 256 = 32v \quad \Rightarrow \quad v = \frac{256}{32} = 8 \]

It is easy to see that \( B'(v) > 0 \) for all \( v < 8 \) and \( B'(v) < 0 \) for all \( v > 8 \). Therefore the maximum value of the area must occur at \( v = 8 \) and this area is \( A = \sqrt{B} = \sqrt{256 \cdot 8 - 16 \cdot 8^2} = 32 \). The dimensions of the rectangle are \( y = \sqrt{8} = 2\sqrt{2} \) and \( x = \sqrt{u} = \sqrt{16 - v} = \sqrt{16 - 8} = \sqrt{8} = 2\sqrt{2} \). So, this rectangle is a square.

EXAMPLE 10: Determine the points on \( y = x^2 + 1 \) that are closest to \((0, 2)\).
EXAMPLE 10: Determine the points on \( y = x^2 + 1 \) that are closest to \((0, 2)\).

Solution: We first draw a picture:

The next step is to create a corresponding mathematical model:

\[
\text{Minimize: } d = \sqrt{(x - 0)^2 + (y - 2)^2} = \sqrt{x^2 + (y - 2)^2}
\]

\[
\text{Constraint: } y = x^2 + 1
\]

We can now substitute \( y = x^2 + 1 \) into the first equation to express \( d \) as a function of one variable. However, this approach involves roots which makes the algebra a bit complicated.

Instead, we square both sides of \( d = \sqrt{x^2 + (y - 2)^2} \). Note that values of \( x \) and \( y \) that minimize \( d = \sqrt{x^2 + (y - 2)^2} \) will also minimize \( d^2 = x^2 + (y - 2)^2 \) and vice-versa. Putting \( D = d^2 \), we can reformulate our problem in the following way:

\[
\text{Minimize: } D = x^2 + (y - 2)^2
\]

\[
\text{Constraint: } y = x^2 + 1
\]

We now solve the second equation for \( x^2 \) and substitute the result into the first equation to express \( D \) as a function of one variable:

\[
y = x^2 + 1 \implies x^2 = y - 1 \implies D = x^2 + (y - 2)^2 = y - 1 + (y - 2)^2 = y^2 - 3y + 3
\]

Since there is no upper bound for \( y \), we can use only the First Derivative Test for Absolute Extreme Values to find the absolute minimum value of \( D = y^2 - 3y + 3 \). The derivative of \( D(y) \) is \( D'(y) = 2y - 3 \), so to find the critical numbers we solve the equation

\[
2y - 3 = 0 \implies y = \frac{3}{2}
\]

It is easy to see that \( D'(y) < 0 \) for all \( y < \frac{3}{2} \) and \( D'(y) > 0 \) for all \( y > \frac{3}{2} \). Therefore the minimum value of the distance must occur at \( y = \frac{3}{2} \) and this distance is

\[
d = \sqrt{D} = \sqrt{\left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2} + 3} = \frac{2}{2} = \frac{1}{\sqrt{2}}
\]

The corresponding \( x \)-coordinates are

\[
x^2 = y - 1 \implies x = \pm \sqrt{y - 1} = \pm \sqrt{\frac{3}{2} - 1} = \pm \frac{1}{\sqrt{2}}
\]

Thus, the points are \( \left(-\frac{1}{\sqrt{2}}, \frac{3}{2}\right) \) and \( \left(\frac{1}{\sqrt{2}}, \frac{3}{2}\right) \).
EXAMPLE 11: A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could proceed in any of three ways:

1. Row his boat directly across the river to point C and then run to B
2. Row directly to B
3. Row to some point D between C and B and then run to B

If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?

Solution: If we let $x$ be the distance from C to D, then the running distance is $|DB| = 8 - x$ and the Pythagorean Theorem gives the rowing distance as $|AD| = \sqrt{x^2 + 9}$. We use the equation

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

Then the rowing time is $\frac{\sqrt{x^2 + 9}}{6}$ and the running time is $(8 - x)/8$, so the total time $T$ as a function of $x$ is

$$T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$$

The domain of this function $T$ is $[0, 8]$. Notice that if $x = 0$ he rows to C and if $x = 8$ he rows directly to B. The derivative of $T$ is

$$T'(x) = \left(\frac{(x^2 + 9)^{1/2}}{6} + \frac{8 - x}{8}\right)' = \frac{1}{6} \left((x^2 + 9)^{1/2}\right)' + \frac{1}{8} (8 - x)'$$

$$= \frac{1}{6} \cdot \frac{1}{2} (x^2 + 9)^{-1/2} \cdot (x^2 + 9)' + \frac{1}{8} (8 - x)'$$

$$= \frac{1}{6} \cdot \frac{1}{2} (x^2 + 9)^{-1/2} \cdot 2x + \frac{1}{8} \cdot (-1)$$

$$= \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

Thus, using the fact that $x \geq 0$, we have

$$T'(x) = 0 \iff \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \iff 4x = 3\sqrt{x^2 + 9} \iff 16x^2 = 9(x^2 + 9) \iff 7x^2 = 81$$

so the only critical point is $9/\sqrt{7}$. To see whether the minimum occurs at this critical number or at an endpoint of the domain $[0, 8]$, we evaluate $T$ at all three points:

$$T(0) = 1.5 \quad T\left(\frac{9}{\sqrt{7}}\right) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \quad T(8) = \frac{\sqrt{73}}{6} \approx 1.42$$

Since the smallest of these values of $T$ occurs when $x = 9/\sqrt{7}$, the absolute minimum value of $T$ must occur there. Thus the man should land the boat at a point $9/\sqrt{7}$ km ($\approx 3.4$ km) downstream from his starting point.
Applications to Business and Economics

DEFINITION: The cost function \( C(x) \) is the cost of producing \( x \) units of a certain product. The marginal cost \( C'(x) \) is the rate of change of \( C \) with respect to \( x \). The demand function (or price function) \( p(x) \) is the price per unit that the company can charge if it sells \( x \) units.

If \( x \) units are sold and the price per unit is \( p(x) \), then the total revenue is

\[
R(x) = xp(x)
\]

and \( R \) is called the revenue function. The derivative \( R' \) of the revenue function is called the marginal revenue function and is the rate of change of revenue with the respect to the number of units sold. If \( x \) units are sold, then the total profit is

\[
P(x) = R(x) - C(x)
\]

and \( P \) is called the profit function. The marginal profit function is \( P' \), the derivative of the profit function.

EXAMPLE: A store has been selling 200 DVD burners a week at $350 each. A market survey indicates that for each $10 rebate offered to buyers, the number of units sold will increase by 20 a week. Find the demand function and the revenue function. How large a rebate should the store offer to maximize its revenue?

Solution: If \( x \) is the number of DVD burners sold per week, then the weekly increase in sales is \( x - 200 \). For each increase of 20 units sold, the price is decreased by $10. So for each additional unit sold, the decrease in price will be \( \frac{1}{20} \times 10 = \frac{1}{2} \) and the demand function is

\[
p(x) = 350 - \frac{1}{2}(x - 200) = 350 - \frac{1}{2}x + \frac{1}{2} \cdot 200 = 350 - \frac{1}{2}x + 100 = 450 - \frac{1}{2}x
\]

The revenue function is

\[
R(x) = xp(x) = 450x - \frac{1}{2}x^2
\]

Since

\[
R'(x) = \left(450x - \frac{1}{2}x^2\right)' = 450x' - \frac{1}{2}(x^2)' = 450 \cdot 1 - \frac{1}{2} \cdot 2x = 450 - x
\]

we see that \( R'(x) = 0 \) when \( x = 450 \). This value of \( x \) gives an absolute maximum by the First Derivative Test (or simply by observing that the graph of \( R \) is a parabola that opens downward). The corresponding price is

\[
p(450) = 450 - \frac{1}{2} \cdot 450 = 450 - 225 = 225
\]

and the rebate is \( 350 - 225 = 125 \). Therefore, to maximize revenue the store should offer a rebate of $125.
Appendix

FOR EXAMPLE 3: We have

\[ C'(w) = \left(60w^2 + \frac{800}{w}\right)' = (60w^2 + 800w^{-1})' = 60(w^2)' + 800(w^{-1})' \]
\[ = 60 \cdot 2w + 800 \cdot (-1)w^{-1-1} \]
\[ = 120w - 800w^{-2} \]
\[ = \frac{120w}{w^2} - \frac{800}{w^2} \]
\[ = \frac{120w^3 - 800}{w^2} \]

FOR EXAMPLE 5: We have

\[ A'(r) = \left(2\pi r^2 + \frac{3000}{r}\right)' = (2\pi r^2 + 3000r^{-1})' = 2\pi(r^2)' + 3000(r^{-1})' \]
\[ = 2\pi \cdot 2r + 3000 \cdot (-1)r^{-1-1} \]
\[ = 4\pi r - 3000r^{-2} \]
\[ = \frac{4\pi r}{r^2} - \frac{3000}{r^2} \]
\[ = \frac{4\pi r^3 - 3000}{r^2} \]

FOR EXAMPLE 7: We have

\[ A'(w) = \left(207 - 3.5w - \frac{400}{w}\right)' = (207 - 3.5w - 400w^{-1})' = 207' - 3.5w' - 400(w^{-1})' \]
\[ = 0 - 3.5 \cdot 1 - 400 \cdot (-1)w^{-1-1} \]
\[ = -3.5 + 400w^{-2} \]
\[ = \frac{-3.5}{w^2} + \frac{400}{w^2} \]
\[ = \frac{-3.5w^2 + 400}{w^2} \]