Curve Sketching

GUIDELINES FOR SKETCHING A CURVE:

A. Domain.

B. Intercepts: x- and y-intercepts.

C. Symmetry: even ($f(-x) = f(x)$) or odd ($f(-x) = -f(x)$) function or neither, periodic function.

D. Asymptotes: horizontal $\left(\lim\limits_{x\to\pm\infty} f(x) = L\right)$ and vertical $\left(\lim\limits_{x\to a^\pm} f(x) = \pm\infty\right)$ asymptotes.

E. Intervals of Increase or Decrease: Use the I/D Test (first derivative).

F. Local Maximum and Minimum Values: Use critical numbers.

G. Concavity and Points of Inflection: Compute $f''(x)$ and use the Concavity Test.

H. Sketch the Curve: Using the information in items A-G, draw the graph.

EXAMPLE: Use the guidelines to sketch the curve $f(x) = \dfrac{x-1}{x+2}$.

Solution:

A. Domain: $\boxed{(-\infty,-2)\cup(-2,\infty)}$

B. Intercepts:

(i) x-intercept ($y=0$):

$$\frac{x-1}{x+2} = 0 \quad\Longrightarrow\quad x-1=0 \quad\Longrightarrow\quad x=1 \quad\Longrightarrow\quad \boxed{(1,0)}$$

(ii) y-intercept ($x=0$):

$$y = \frac{0-1}{0+2} = -\frac{1}{2} \quad\Longrightarrow\quad \boxed{\left(0,-\frac{1}{2}\right)}$$

C. Symmetry:

(i) This function is $\boxed{\text{not even}}$, since

$$f(-1) = \frac{-1-1}{-1+2} \neq \frac{1-1}{1+2} = f(1)$$

(ii) This function is $\boxed{\text{not odd}}$, since

$$f(-1) = \frac{-1-1}{-1+2} \neq -\frac{1-1}{1+2} = -f(1)$$

(iii) This function is $\boxed{\text{not periodic}}$.

D. Asymptotes:

(i) Horizontal asymptote:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x-1}{x+2} = \lim_{x\to\pm\infty} \frac{\frac{x-1}{x}}{\frac{x+2}{x}}$$

$$= \lim_{x\to\pm\infty} \frac{1-\frac{1}{x}}{1+\frac{2}{x}} = 1 \quad \Longrightarrow \quad \boxed{y = 1 \text{ is the horizontal asymptote}}$$

(ii) Vertical asymptote:

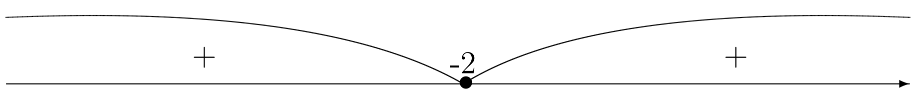
$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x-1}{x+2} = \left[\begin{array}{l} \text{WORK:} \\ \dfrac{-2.01-1}{-2.01+2} = \dfrac{-3.01}{-0.01} = \dfrac{-\text{“NOT SMALL”}}{-\text{“SMALL”}} = +\text{“BIG”} \end{array}\right] = \infty$$

Therefore $\boxed{x = -2 \text{ is the vertical asymptote}}$.

E. Intervals of Increase or Decrease: We have

$$f'(x) = \left(\frac{x-1}{x+2}\right)' = \frac{(x-1)'(x+2)-(x-1)(x+2)'}{(x+2)^2} = \frac{1\cdot(x+2)-(x-1)\cdot 1}{(x+2)^2}$$

$$= \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2} > 0$$

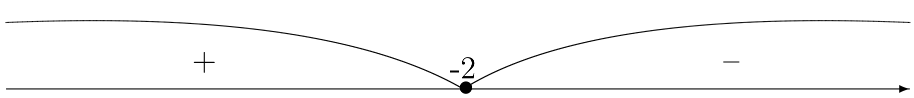


It follows that this function $\boxed{\text{increases on } (-\infty,-2) \text{ and } (-2,\infty)}$.

F. Local Maximum and Minimum Values: This function has no critical numbers, therefore it has $\boxed{\text{no local maximum and minimum values}}$.

G. Concavity and Points of Inflection: We have

$$f''(x) = \left(\frac{3}{(x+2)^2}\right)' = \left(3(x+2)^{-2}\right)' = 3\left((x+2)^{-2}\right)' = -6(x+2)^{-3} = -\frac{6}{(x+2)^3}$$



It follows that $f''(x) > 0$ if $x < -2$ and $f''(x) < 0$ if $x > -2$, therefore $f(x)$ is

$$\boxed{\text{concave upward on } (-\infty,-2)} \text{ and } \boxed{\text{concave downward on } (-2,\infty)}$$

There are $\boxed{\text{no inflection points}}$, since $x = -2$ is not in the domain.

H. Sketch the Curve:

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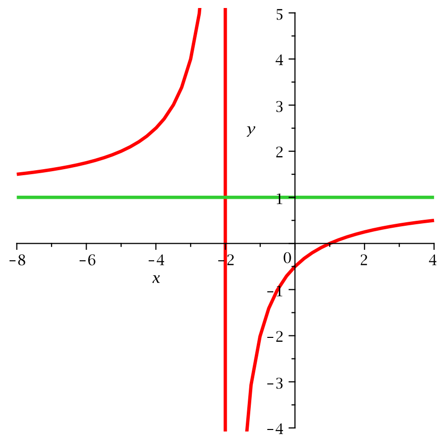


EXAMPLE: Use the guidelines to sketch the curve $f(x)=\dfrac{2x^2}{x^2-1}$.

Solution:

A. Domain:

B. Intercepts:

(i) x-intercepts:

(ii) y-intercepts:

C. Symmetry:

(i) This function is even (?)

(ii) This function is odd (?)

(iii) This function is periodic (?)

D. Asymptotes:

(i) Horizontal asymptotes:

(ii) Vertical asymptotes:

E. Intervals of Increase or Decrease:

F. Local Maximum and Minimum Values:

G. Concavity and Points of Inflection:

H. Sketch the Curve:

EXAMPLE: Use the guidelines to sketch the curve $f(x) = \dfrac{2x^2}{x^2 - 1}$.

Solution:

A. Domain: $\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$

B. Intercepts:

(i) x-intercept ($y = 0$):

$$\frac{2x^2}{x^2-1} = 0 \quad \Longrightarrow \quad 2x^2 = 0 \quad \Longrightarrow \quad x = 0 \quad \Longrightarrow \quad \boxed{(0,0)}$$

(ii) y-intercept ($x = 0$):

$$y = \frac{2 \cdot 0^2}{0^2 - 1} = 0 \quad \Longrightarrow \quad \boxed{(0,0)}$$

C. Symmetry:

(i) This function is $\boxed{\text{even}}$, since

$$f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2-1} = f(x)$$

(ii) This function is $\boxed{\text{not odd}}$, since it is even.

(iii) This function is $\boxed{\text{not periodic}}$.

D. Asymptotes:

(i) Horizontal asymptote:

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-1} = \lim_{x\to\pm\infty} \frac{\frac{2x^2}{x^2}}{\frac{x^2-1}{x^2}} = \lim_{x\to\pm\infty} \frac{\frac{2x^2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$

$$= \lim_{x\to\pm\infty} \frac{2}{1 - \frac{1}{x^2}} = 2 \quad \Longrightarrow \quad \boxed{y = 2 \text{ is the horizontal asymptote}}$$

(ii) Vertical asymptotes:

$$\lim_{x\to 1^+} \frac{2x^2}{x^2-1} = \left[\begin{array}{l}\text{WORK:} \\ \dfrac{2(1.01)^2}{(1.01)^2 - 1} = \dfrac{\approx 2}{\approx 0.02} = \dfrac{+\text{“NOT SMALL”}}{+\text{“SMALL”}} = +\text{“BIG”}\end{array}\right] = \infty$$

so $x = 1$ is the first vertical asymptote. Similarly,

$$\lim_{x\to -1^-} \frac{2x^2}{x^2-1} = \left[\begin{array}{l}\text{WORK:} \\ \dfrac{2(-1.01)^2}{(-1.01)^2 - 1} = \dfrac{\approx 2}{\approx 0.02} = \dfrac{+\text{“NOT SMALL”}}{+\text{“SMALL”}} = +\text{“BIG”}\end{array}\right] = \infty$$

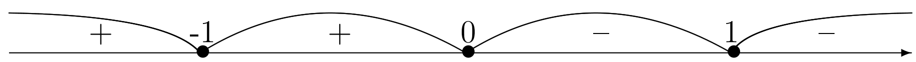
so $x = -1$ is the second vertical asymptote.

E. Intervals of Increase or Decrease: We have

$$f'(x) = \left(\frac{2x^2}{x^2-1}\right)' = \frac{(2x^2)'(x^2-1) - 2x^2(x^2-1)'}{(x^2-1)^2} = \frac{4x(x^2-1) - 2x^2 \cdot 2x}{(x^2-1)^2}$$

$$= \frac{4x^3 - 4x - 4x^3}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$



It follows that this function

increases on $(-\infty, -1)$ and $(-1, 0)$ and **decreases on $(0, 1)$ and $(1, \infty)$**

F. Local Maximum and Minimum Values: **$x = 0$ is the point of local maximum**.

G. Concavity and Points of Inflection: We have

$$f''(x) = \left(\frac{-4x}{(x^2-1)^2}\right)' = \frac{(-4x)'(x^2-1)^2 - (-4x)((x^2-1)^2)'}{((x^2-1)^2)^2}$$

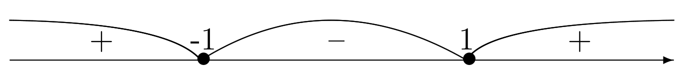
$$= \frac{-4(x^2-1)^2 - (-4x) \cdot 2(x^2-1)^{2-1} \cdot (x^2-1)'}{(x^2-1)^4}$$

$$= \frac{-4(x^2-1)^2 - (-4x) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4}$$

$$= \frac{-4(x^2-1) - (-4x) \cdot 2 \cdot 2x}{(x^2-1)^3}$$

$$= \frac{-4x^2 + 4 + 16x^2}{(x^2-1)^3}$$

$$= \frac{12x^2 + 4}{(x^2-1)^3}$$



It follows that $f''(x) > 0$ if $x < -1$ or $x > 1$ and $f''(x) < 0$ if $-1 < x < 1$. Thus $f(x)$ is

concave upward on $(-\infty, -1)$ and $(1, \infty)$ and **concave downward on $(-1, 1)$**

There are **no inflection points**, since $x = \pm 1$ are not in the domain.

H. Sketch the Curve:

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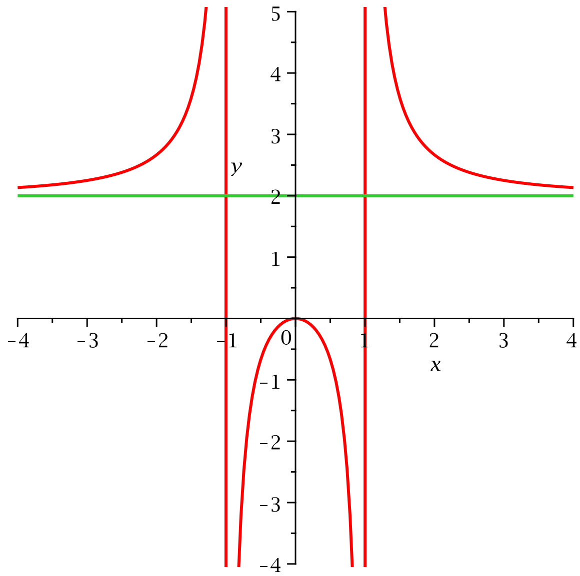


EXAMPLE: Use the guidelines to sketch the curve $f(x) = xe^{-x^2/2}$.

Solution:

A. Domain:

B. Intercepts:

(i) x-intercepts:

(ii) y-intercepts:

C. Symmetry:

(i) This function is even (?)

(ii) This function is odd (?)

(iii) This function is periodic (?)

D. Asymptotes:

(i) Horizontal asymptotes:

(ii) Vertical asymptotes:

E. Intervals of Increase or Decrease:

F. Local Maximum and Minimum Values:

G. Concavity and Points of Inflection:

H. Sketch the Curve:

EXAMPLE: Use the guidelines to sketch the curve $f(x) = xe^{-x^2/2}$.

Solution:

A. Domain: $\boxed{(-\infty, \infty)}$

B. Intercepts:

(i) x-intercept ($y = 0$):

$$xe^{-x^2/2} = 0 \quad \Longrightarrow \quad x = 0 \quad \Longrightarrow \quad \boxed{(0,0)}$$

(ii) y-intercept ($x = 0$):

$$y = 0 \cdot e^{-0^2/2} = 0 \quad \Longrightarrow \quad \boxed{(0,0)}$$

C. Symmetry:

(i) This function is $\boxed{\text{not even}}$, since

$$f(-1) = (-1)e^{-(-1)^2/2} \neq 1 \cdot e^{-1^2/2} = f(1)$$

(ii) This function is $\boxed{\text{odd}}$, since

$$f(-x) = (-x)e^{-(-x)^2/2} = -xe^{-x^2/2} = -f(x)$$

(iii) This function is $\boxed{\text{not periodic}}$.

D. Asymptotes:

(i) Horizontal asymptote:

$$\lim_{x\to\pm\infty} xe^{-x^2/2} = \lim_{x\to\pm\infty} \frac{x}{e^{x^2/2}} = \lim_{x\to\pm\infty} \frac{x'}{(e^{x^2/2})'} = \lim_{x\to\pm\infty} \frac{1}{e^{x^2/2}(x^2/2)'} = \lim_{x\to\pm\infty} \frac{1}{e^{x^2/2} \cdot x} = 0$$

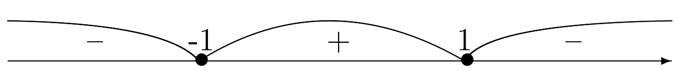
Therefore $\boxed{y = 0 \text{ is the horizontal asymptote}}$.

(ii) $\boxed{\text{No vertical asymptotes}}$.

E. Intervals of Increase or Decrease: We have

$$f'(x) = \left(xe^{-x^2/2}\right)' = x'e^{-x^2/2} + x(e^{-x^2/2})' = e^{-x^2/2} + xe^{-x^2/2} \cdot \left(-\frac{x^2}{2}\right)'$$

$$= e^{-x^2/2} + xe^{-x^2/2} \cdot (-x) = (1 - x^2)e^{-x^2/2}$$



One can see that this function

$$\boxed{\text{increases on } \ (1, -1)} \text{ and } \boxed{\text{decreases on } \ (-\infty, -1) \text{ and } (1, \infty)}$$

F. Local Maximum and Minimum Values:

$$\boxed{x = -1 \ \text{ is local minimum}} \text{ and } \boxed{x = 1 \ \text{ is local maximum}}$$

G. Concavity and Points of Inflection: We have

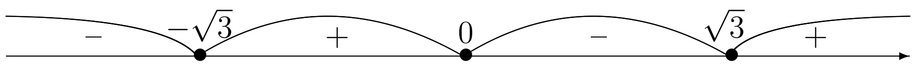
$$f''(x) = \left((1-x^2)e^{-x^2/2}\right)' = (1-x^2)'e^{-x^2/2} + (1-x^2)\left(e^{-x^2/2}\right)'$$

$$= -2xe^{-x^2/2} + (1-x^2)e^{-x^2/2} \cdot \left(-\frac{x^2}{2}\right)'$$

$$= -2xe^{-x^2/2} - x(1-x^2)e^{-x^2/2}$$

$$= (-2-1+x^2)xe^{-x^2/2}$$

$$= (x^2-3)xe^{-x^2/2}$$



It follows that $f''(x) > 0$ if $-\sqrt{3} < x < 0$ or $x > \sqrt{3}$ and $f''(x) < 0$ if $x < -\sqrt{3}$ or $0 < x < \sqrt{3}$. Thus $f(x)$ is

$$\boxed{\text{concave upward on } \ (-\sqrt{3}, 0) \ \text{ and } \ (\sqrt{3}, \infty)}$$

and

$$\boxed{\text{concave downward on } \ (-\infty, -\sqrt{3}) \ \text{ and } \ (0, \sqrt{3})}$$

There are $\boxed{\text{three inflection points } x = 0, \pm\sqrt{3}}$.