Curve Sketching

GUIDELINES FOR SKETCHING A CURVE:

A. Domain.

B. Intercepts: $x$- and $y$-intercepts.

C. Symmetry: even ($f(-x) = f(x)$) or odd ($f(-x) = -f(x)$) function or neither, periodic function.

D. Asymptotes: horizontal ($\lim_{x \to \pm \infty} f(x) = L$) and vertical ($\lim_{x \to a^\pm} f(x) = \pm \infty$) asymptotes.

E. Intervals of Increase or Decrease: Use the I/D Test (first derivative).

F. Local Maximum and Minimum Values: Use critical numbers.

G. Concavity and Points of Inflection: Compute $f''(x)$ and use the Concavity Test.

H. Sketch the Curve: Using the information in items A-G, draw the graph.

EXAMPLE: Use the guidelines to sketch the curve $f(x) = \frac{x - 1}{x + 2}$.

Solution:

A. Domain: $(-\infty, -2) \cup (-2, \infty)$

B. Intercepts:
   (i) $x$-intercept ($y = 0$):
   \[
   \frac{x - 1}{x + 2} = 0 \quad \Rightarrow \quad x - 1 = 0 \quad \Rightarrow \quad x = 1 \quad \Rightarrow \quad (1,0)
   \]
   (ii) $y$-intercept ($x = 0$):
   \[
   y = \frac{0 - 1}{0 + 2} = -\frac{1}{2} \quad \Rightarrow \quad \left(0, -\frac{1}{2}\right)
   \]

C. Symmetry:
   (i) This function is not even, since
   \[
   f(-1) = \frac{-1 - 1}{-1 + 2} = \frac{1 - 1}{1 + 2} = f(1)
   \]
   (ii) This function is not odd, since
   \[
   f(-1) = \frac{-1 - 1}{-1 + 2} \neq \frac{1 - 1}{1 + 2} = -f(1)
   \]
   (iii) This function is not periodic.
D. Asymptotes:

(i) Horizontal asymptote:

\[ \lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{x-1}{x+2} = \lim_{x \to \pm \infty} \frac{\frac{x-1}{x}}{\frac{x+2}{x}} = \lim_{x \to \pm \infty} \frac{1-\frac{1}{x}}{1+\frac{2}{x}} = 1 \implies y = 1 \text{ is the horizontal asymptote} \]

(ii) Vertical asymptote:

\[ \lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x-1}{x+2} = \left[ \begin{array}{c} \text{WORK:} \\
-2.01-1 \\
-2.01+2 \\
-0.01 \\
\end{array} \right] = -3.01 = \frac{\text{"NOT SMALL"}}{-\text{"SMALL"}} = +\text{"BIG"} \]

Therefore \( x = -2 \) is the vertical asymptote.

E. Intervals of Increase or Decrease: We have

\[ f'(x) = \left( \frac{x-1}{x+2} \right)' = \frac{(x-1)’(x+2) - (x-1)(x+2)’}{(x+2)^2} = \frac{1 \cdot (x+2) - (x-1) \cdot 1}{(x+2)^2} = \frac{x+2-x+1}{(x+2)^2} = \frac{3}{(x+2)^2} > 0 \]

It follows that this function increases on \((-\infty, -2)\) and \((-2, \infty)\).

F. Local Maximum and Minimum Values: This function has no critical numbers, therefore it has no local maximum and minimum values.

G. Concavity and Points of Inflection: We have

\[ f''(x) = \left( \frac{3}{(x+2)^2} \right)' = (3(x+2)^{-2})' = 3 ((x+2)^{-2})' = -6(x+2)^{-3} = -\frac{6}{(x+2)^3} \]

It follows that \( f''(x) > 0 \) if \( x < -2 \) and \( f''(x) < 0 \) if \( x > -2 \), therefore \( f(x) \) is concave upward on \((-\infty, -2)\) and concave downward on \((-2, \infty)\).

There are no inflection points, since \( x = -2 \) is not in the domain.

H. Sketch the Curve:
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EXAMPLE: Use the guidelines to sketch the curve $f(x) = \frac{2x^2}{x^2 - 1}$.

Solution:

A. Domain:

B. Intercepts:
   (i) $x$-intercepts:
   (ii) $y$-intercepts:

C. Symmetry:
   (i) This function is even (?)
   (ii) This function is odd (?)
   (iii) This function is periodic (?)
D. Asymptotes:

(i) Horizontal asymptotes:

(ii) Vertical asymptotes:

E. Intervals of Increase or Decrease:
F. Local Maximum and Minimum Values:

G. Concavity and Points of Inflection:

H. Sketch the Curve:
EXAMPLE: Use the guidelines to sketch the curve \( f(x) = \frac{2x^2}{x^2 - 1} \).

Solution:

A. Domain: \((−∞, −1) \cup (−1, 1) \cup (1, ∞)\)

B. Intercepts:

(i) \(x\)-intercept \((y = 0)\):
\[
\frac{2x^2}{x^2 - 1} = 0 \implies 2x^2 = 0 \implies x = 0 \implies (0, 0)
\]

(ii) \(y\)-intercept \((x = 0)\):
\[
y = \frac{2 \cdot 0^2}{0^2 - 1} = 0 \implies (0, 0)
\]

C. Symmetry:

(i) This function is \(\text{even}\), since
\[
f(−x) = \frac{2(−x)^2}{(−x)^2 - 1} = \frac{2x^2}{x^2 - 1} = f(x)
\]

(ii) This function is \(\text{not odd}\) since it is even.

(iii) This function is \(\text{not periodic}\).

D. Asymptotes:

(i) Horizontal asymptote:
\[
\lim_{x \to \pm \infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm \infty} \frac{2x^2}{x^2} = \lim_{x \to \pm \infty} \frac{2x^2}{x^2} - \frac{1}{x^2}
\]
\[
= \lim_{x \to \pm \infty} \frac{2}{1 - \frac{1}{x^2}} = 2 \implies y = 2 \text{ is the horizontal asymptote}
\]

(ii) Vertical asymptotes:
\[
\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \frac{2(1.01)^2}{(1.01)^2 - 1} \approx 2 = \text{“NOT SMALL”}
\]
\[
\lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = \frac{2(-1.01)^2}{(-1.01)^2 - 1} \approx 2 = \text{“NOT SMALL”}
\]
so \(x = 1\) is the first vertical asymptote. Similarly,
\[
\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = \frac{2(1.01)^2}{(1.01)^2 - 1} \approx 2 = \text{“NOT SMALL”}
\]
\[
\lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \frac{2(-1.01)^2}{(-1.01)^2 - 1} \approx 2 = \text{“NOT SMALL”}
\]
so \(x = -1\) is the second vertical asymptote.
E. Intervals of Increase or Decrease: We have

\[
f'(x) = \left( \frac{2x^2}{x^2 - 1} \right)' = \frac{(2x^2)'(x^2 - 1) - 2x^2(x^2 - 1)'}{(x^2 - 1)^2} = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{4x^3 - 4x - 4x^3}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}
\]

It follows that this function increases on \((-\infty, -1)\) and \((-1, 0)\) and decreases on \((0, 1)\) and \((1, \infty)\).

F. Local Maximum and Minimum Values: \(x = 0\) is the point of local maximum.

G. Concavity and Points of Inflection: We have

\[
f''(x) = \left( \frac{-4x}{(x^2 - 1)^2} \right)' = \frac{(-4x)'(x^2 - 1)^2 - (-4x)((x^2 - 1)^2)'}{(x^2 - 1)^4} = \frac{-4(x^2 - 1)^2 - (-4x) \cdot 2(x^2 - 1)^2 - 1 \cdot (x^2 - 1)'}{(x^2 - 1)^4} = \frac{-4(x^2 - 1)^2 - (-4x) \cdot 2(x^2 - 1) \cdot 2x}{(x^2 - 1)^4} = \frac{-4(x^2 - 1) - (-4x) \cdot 2 \cdot 2x}{(x^2 - 1)^3} = \frac{-4x^2 + 4 + 16x^2}{(x^2 - 1)^3} = \frac{12x^2 + 4}{(x^2 - 1)^3}
\]

It follows that \(f''(x) > 0\) if \(x < -1\) or \(x > 1\) and \(f''(x) < 0\) if \(-1 < x < 1\). Thus \(f(x)\) is concave upward on \((-\infty, -1)\) and \((1, \infty)\) and concave downward on \((-1, 1)\).

There are no inflection points, since \(x = \pm 1\) are not in the domain.

H. Sketch the Curve:
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EXAMPLE: Use the guidelines to sketch the curve \( f(x) = xe^{-x^2/2} \).

Solution:

A. Domain:

B. Intercepts:

(i) \( x \)-intercepts:

(ii) \( y \)-intercepts:

C. Symmetry:

(i) This function is even (?)

(ii) This function is odd (?)

(iii) This function is periodic (?)
D. Asymptotes:
   (i) Horizontal asymptotes:

   (ii) Vertical asymptotes:

E. Intervals of Increase or Decrease:

F. Local Maximum and Minimum Values:
G. Concavity and Points of Inflection:

H. Sketch the Curve:
EXAMPLE: Use the guidelines to sketch the curve \( f(x) = xe^{-x^2/2} \).

Solution:

A. Domain: \((-\infty, \infty)\)

B. Intercepts:

(i) \( x \)-intercept \((y = 0)\):
\[
x e^{-x^2/2} = 0 \implies x = 0 \implies (0,0)
\]

(ii) \( y \)-intercept \((x = 0)\):
\[
y = 0 \cdot e^{-0^2/2} = 0 \implies (0,0)
\]

C. Symmetry:

(i) This function is not even, since
\[
f(-1) = (-1)e^{-(1)^2/2} \neq 1 \cdot e^{-1^2/2} = f(1)
\]

(ii) This function is odd since
\[
f(-x) = (-x)e^{-(x)^2/2} = -xe^{-x^2/2} = -f(x)
\]

(iii) This function is not periodic.

D. Asymptotes:

(i) Horizontal asymptote:
\[
\lim_{x \to \pm \infty} xe^{-x^2/2} = \lim_{x \to \pm \infty} \frac{x}{e^{x^2/2}} = \lim_{x \to \pm \infty} \frac{x'}{e^{x^2/2}(x^2/2)'} = \lim_{x \to \pm \infty} \frac{1}{e^{x^2/2} \cdot x} = 0
\]
Therefore \(y = 0\) is the horizontal asymptote.

(ii) No vertical asymptotes.

E. Intervals of Increase or Decrease: We have
\[
f'(x) = \left(x e^{-x^2/2}\right)' = x' e^{-x^2/2} + x(e^{-x^2/2})' = e^{-x^2/2} + xe^{-x^2/2} \cdot \left(-\frac{x^2}{2}\right)'
\]
\[
= e^{-x^2/2} + xe^{-x^2/2} \cdot (-x) = (1 - x^2)e^{-x^2/2}
\]

One can see that this function increases on \((1, -1)\) and decreases on \((-\infty, -1)\) and \((1, \infty)\).

F. Local Maximum and Minimum Values:
\[
x = -1 \text{ is local minimum} \text{ and } x = 1 \text{ is local maximum}
\]
G. Concavity and Points of Inflection: We have

\[ f''(x) = \left( (1 - x^2)e^{-x^2/2} \right)' = (1 - x^2)'e^{-x^2/2} + (1 - x^2) \left( e^{-x^2/2} \right)' \]

\[ = -2xe^{-x^2/2} + (1 - x^2)e^{-x^2/2} \cdot \left( \frac{-x^2}{2} \right)' \]

\[ = -2xe^{-x^2/2} - x(1 - x^2)e^{-x^2/2} \]

\[ = (-2 - 1 + x^2)xe^{-x^2/2} \]

\[ = (x^2 - 3)xe^{-x^2/2} \]

It follows that \( f''(x) > 0 \) if \(-\sqrt{3} < x < 0 \) or \( x > \sqrt{3} \) and \( f''(x) < 0 \) if \( x < -\sqrt{3} \) or \( 0 < x < \sqrt{3} \). Thus \( f(x) \) is

concave upward on \((-\sqrt{3}, 0) \) and \((\sqrt{3}, \infty) \)

and

concave downward on \((-\infty, -\sqrt{3}) \) and \((0, \sqrt{3}) \)

There are three inflection points \( x = 0, \pm \sqrt{3} \).