Derivatives and the Shapes of Graphs

INCREASING/DECREASING TEST:

(a) If $f'(x) > 0$ on an (open) interval I, then f is increasing on I.

(b) If $f'(x) < 0$ on an (open) interval I, then f is decreasing on I.

EXAMPLES:

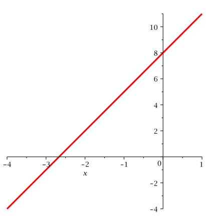
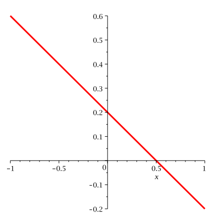
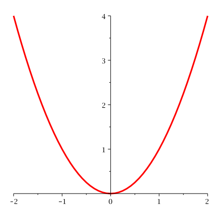
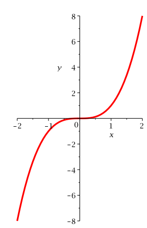
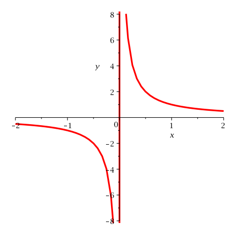
1. The function $f(x) = 3x + 8$ is increasing on $(-\infty, \infty)$, because $f'(x) = 3$ is positive on $(-\infty, \infty)$.

2. The function $f(x) = \dfrac{1-2x}{5}$ is decreasing on $(-\infty, \infty)$, because $f'(x) = -\dfrac{2}{5}$ is negative on $(-\infty, \infty)$.

3. The function $f(x) = x^2$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, because $f'(x) = 2x$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$.

4. The function $f(x) = x^3$ is increasing on $(-\infty, \infty)$, because $f'(x) = 3x^2$ is positive (nonnegative) on $(-\infty, \infty)$.

5. The function $f(x) = \dfrac{1}{x}$ is decreasing on $(-\infty, 0)$ and $(0, \infty)$, because $f'(x) = -\dfrac{1}{x^2}$ is negative everywhere except for $x = 0$.

EXAMPLE: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

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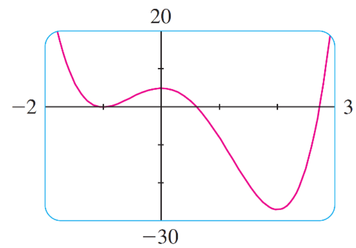
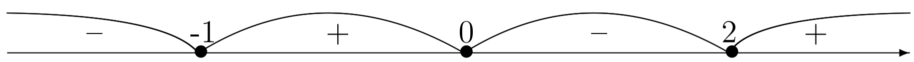
Solution: Since

$$f'(x) = (3x^4 - 4x^3 - 12x^2 + 5)' = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$$

we have

Interval	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$





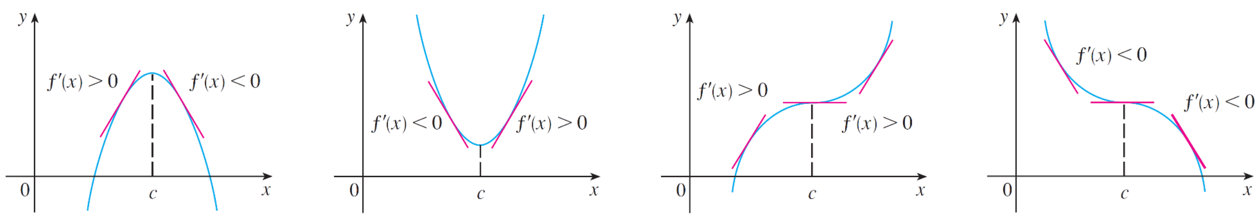
Therefore f is increasing on $(-1, 0)$ and $(2, \infty)$; it is decreasing on $(-\infty, -1)$ and $(0, 2)$.

THE FIRST DERIVATIVE TEST: Suppose c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.



(a) Local maximum (b) Local minimum (c) No maximum or minimum (d) No maximum or minimum

EXAMPLE: Find where f is increasing and where it is decreasing. Find the local maximum and minimum values of f.

(a) $f(x) = x^4 - 4x^3 + 4x^2$ (b) $f(x) = 2x + 3\sqrt[3]{x^2}$

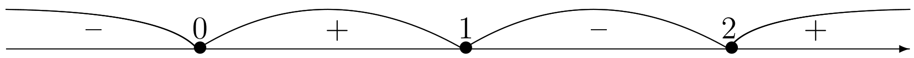
EXAMPLE: Find where f is increasing and where it is decreasing. Find the local maximum and minimum values of f.

(a) $f(x) = x^4 - 4x^3 + 4x^2 = x^2(x^2 - 4x + 4) = x^2(x-2)^2$

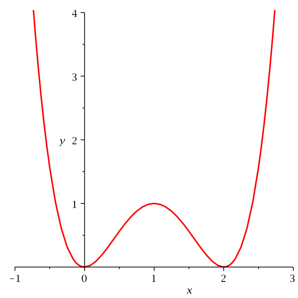
Solution: Since

$$f'(x) = (x^4 - 4x^3 + 4x^2)' = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2)$$

the critical numbers are $x = 0, 1$ and 2. We have



Therefore f is increasing on $(0,1)$ and $(2,\infty)$; it is decreasing on $(-\infty,0)$ and $(1,2)$. Because $f'(x)$ changes from negative to positive at 0 and 2, the First Derivative Test tells us that $f(0) = 0$ and $f(2) = 0$ are local minimum values. Similarly, since $f'(x)$ changes from positive to negative at 1, $f(1) = 1$ is a local maximum value.



(b) $f(x) = 2x + 3\sqrt[3]{x^2}$

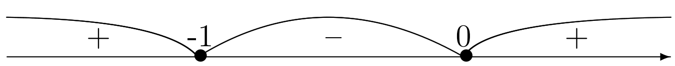
Solution: Since

$$f'(x) = \left(2x + 3x^{2/3}\right)' = 2 + 3 \cdot \frac{2}{3}x^{2/3-1} = 2 + 2x^{-1/3} = 2 + \frac{2}{\sqrt[3]{x}} = \frac{2\sqrt[3]{x}}{\sqrt[3]{x}} + \frac{2}{\sqrt[3]{x}} = \frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

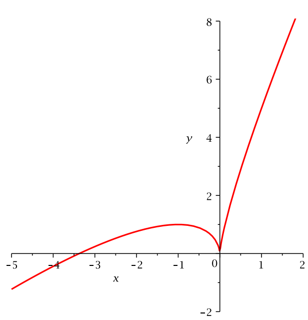
or

$$f'(x) = \left(2x + 3x^{2/3}\right)' = 2 + 3 \cdot \frac{2}{3}x^{2/3-1} = 2 + 2x^{-1/3} = 2x^{-1/3}(x^{1/3}+1) = \frac{2(\sqrt[3]{x}+1)}{\sqrt[3]{x}}$$

the critical numbers are $x = -1$ and 0. We have

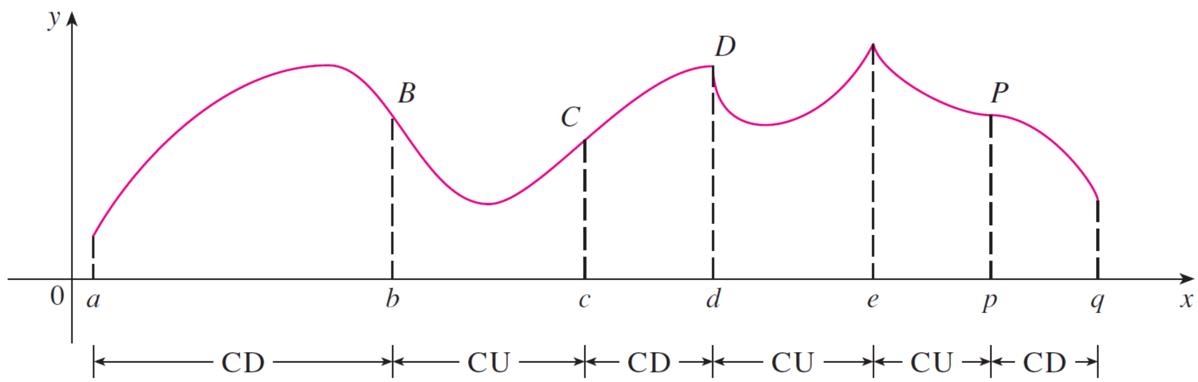


Therefore f is increasing on $(-\infty,-1)$ and $(0,\infty)$; it is decreasing on $(-1,0)$. Because $f'(x)$ changes from positive to negative at -1, the First Derivative Test tells us that $f(-1) = 1$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 0, $f(0) = 0$ is a local minimum value.



DEFINITION: If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on I.

DEFINITION: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



EXAMPLE: In the picture above, B, C, D, and P are the points of inflection.

CONCAVITY TEST:

(a) If $f''(x) > 0$ on an (open) interval I, then the graph of f is concave upward on I.

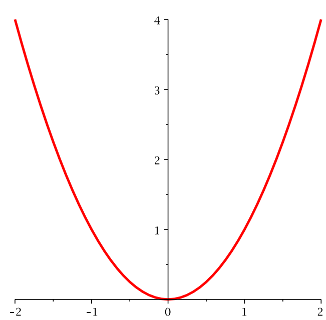
(b) If $f''(x) < 0$ on an (open) interval I, then the graph of f is concave downward on I.

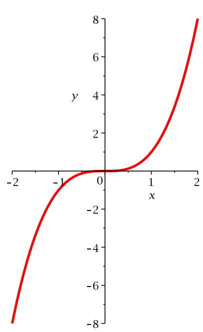
EXAMPLES:

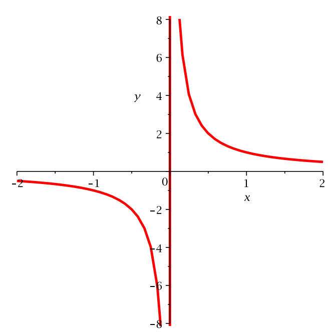
1. The function $f(x) = x^2$ is concave upward on $(-\infty, \infty)$, because $f''(x) = 2$ is positive on $(-\infty, \infty)$. There are no inflection points.

2. The function $f(x) = x^3$ is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$ because $f''(x) = 6x$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. The point $x = 0$ is the inflection point.

3. The function $f(x) = \dfrac{1}{x}$ is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$ because $f''(x) = \dfrac{2}{x^3}$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$. There are no inflection points.







EXAMPLE: Find intervals of concavity and the inflection points of $f(x) = x^4 - 4x^3 + 4x^2$.

EXAMPLE: Find intervals of concavity and the inflection points of $f(x) = x^4 - 4x^3 + 4x^2$.

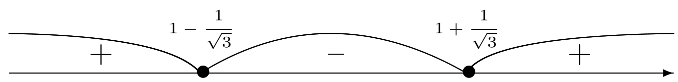
Solution: Since $f'(x) = 4x^3 - 12x^2 + 8x$, we have

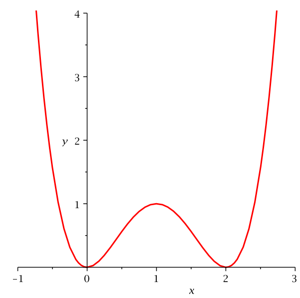
$$f''(x) = (4x^3 - 12x^2 + 8x)' = 12x^2 - 24x + 8 = 4(3x^2 - 6x + 2)$$

Solving the quadratic equation $3x^2 - 6x + 2 = 0$, we obtain that $f''(x) = 0$ at

$$x = \frac{6 \pm \sqrt{6^2 - 4 \cdot 3 \cdot 2}}{2 \cdot 3} = \frac{6 \pm \sqrt{12}}{6} = \left\{ \frac{6}{6} \pm \frac{\sqrt{12}}{6} = 1 \pm \frac{\sqrt{4 \cdot 3}}{6} = 1 \pm \frac{\sqrt{4}\sqrt{3}}{6} = 1 \pm \frac{2\sqrt{3}}{2 \cdot 3} = 1 \pm \frac{\sqrt{3}}{3} = 1 \pm \frac{1 \cdot \sqrt{3}}{\sqrt{3}\sqrt{3}} \right\} = 1 \pm \frac{1}{\sqrt{3}}$$

We have



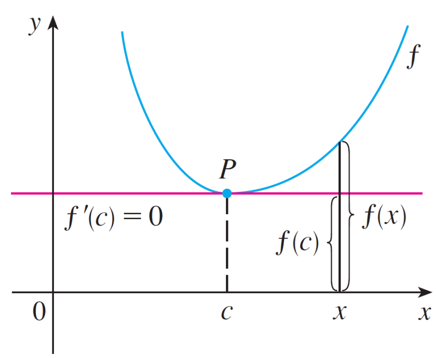


Therefore f is concave upward on $(-\infty, 1 - 1/\sqrt{3})$ and $(1 + 1/\sqrt{3}, \infty)$; it is concave downward on $(1 - 1/\sqrt{3}, 1 + 1/\sqrt{3})$. The points $1 \pm 1/\sqrt{3}$ are the inflection points since the curve changes from concave upward to concave downward and vice-versa there.

THE SECOND DERIVATIVE TEST: Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.



EXAMPLES:

1. Let $f(x) = x^2$. Since $f'(0) = 0$ and $f''(0) = 2 > 0$, it follows that f has a local minimum at 0.

2. Let $f(x) = x$. Since $f'(x) = 1$, the test is inconclusive. Note, that $f(x)$ does not have local extreme values by the First Derivative Test.

3. Let $f(x) = x^4$. Since $f'(0) = 0$ and $f''(0) = 0$, the test is inconclusive. Note, that $f(x)$ has the local minimum at $x = 0$ by the First Derivative Test.

EXAMPLE: Discuss the curve $f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

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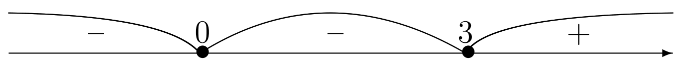
Solution: We have

$$f'(x)=4x^3-12x^2=4x^2(x-3), \quad f''(x)=12x^2-24x=12x(x-2)$$

To find the critical numbers we set $f'(x)=0$ and obtain $x=0$ and $x=3$. To use the Second Derivative Test we evaluate f'' at these critical numbers:

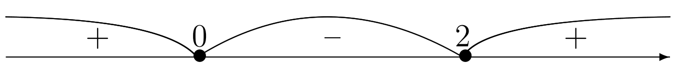
$$f''(0)=0, \quad f''(3)=36>0$$

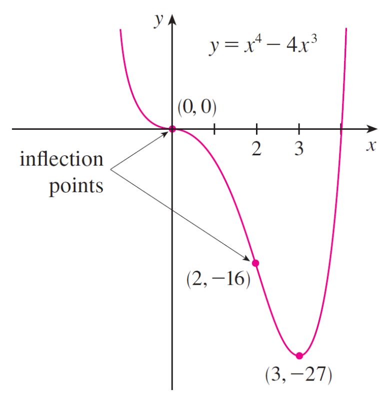
Since $f'(3)=0$ and $f''(3)>0$, $f(3)=-27$ is a local minimum. Since $f''(0)=0$, the Second Derivative Test gives no information about the critical number 0. But since $f'(x)<0$ for $x<0$ and also for $0<x<3$, the First Derivative Test tells us that f does not have a local maximum or minimum at 0:



Since $f''(x)=0$ when $x=0$ or 2, we divide the real line into intervals with these numbers as endpoints and complete the following chart.

Interval	$f''(x)=12x(x-2)$	Concavity
$(-\infty,0)$	$+$	upward
$(0,2)$	$-$	downward
$(2,\infty)$	$+$	upward





The point $(0,0)$ is an inflection point since the curve changes from concave upward to concave downward there. Also, $(2,\ -16)$ is an inflection point since the curve changes from concave downward to concave upward there.

NOTE: The Second Derivative Test is inconclusive when $f''(c)=0$. In other words, at such a point there might be a maximum (for example, $f(x)=-x^4$), there might be a minimum (for example, $f(x)=x^4$), or there might be neither (as in the Example above). This test also fails when $f''(c)$ does not exist. In such cases the First Derivative Test must be used. In fact, even when both tests apply, the First Derivative Test is often the easier one to use.

Appendix

EXAMPLE: Let $f(x) = 3x^2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

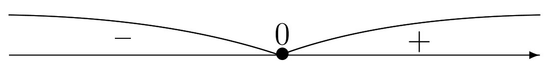
$$f'(x) = (3x^2)' = 3(x^2)' = 3 \cdot 2x = 6x$$

Since

$$6x = 0 \quad \Longleftrightarrow \quad x = 0$$

it follows that the critical number of f is $x = 0$.

(b) We have



Therefore f is increasing on $(0, \infty)$; it is decreasing on $(-\infty, 0)$.

(c) Because $f'(x)$ changes from negative to positive at 0, the First Derivative Test tells us that $f(0) = 0$ is a local minimum value.

(d) We have

$$f''(x) = (6x)' = 6(x)' = 6 \cdot 1 = 6$$

Since $f'(0) = 0$ and $f''(0) = 6 > 0$, the Second Derivative Test tells us that f has a local minimum at 0.

EXAMPLE: Let $f(x) = 7x^2 - 3x + 2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

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(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

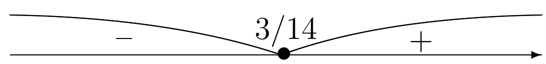
$$f'(x) = (7x^2 - 3x + 2)' = (7x^2)' - (3x)' + 2' = 7(x^2)' - 3(x)' + 2' = 7 \cdot 2x - 3 \cdot 1 + 0 = 14x - 3$$

Since

$$14x - 3 = 0 \quad \iff \quad 14x = 3 \quad \iff \quad x = \frac{3}{14}$$

it follows that the critical number of f is $x = \dfrac{3}{14}$.

(b) We have



Therefore f is increasing on $\left(\dfrac{3}{14}, \infty\right)$; it is decreasing on $\left(-\infty, \dfrac{3}{14}\right)$.

(c) Because $f'(x)$ changes from negative to positive at $\dfrac{3}{14}$, the First Derivative Test tells us that $f\left(\dfrac{3}{14}\right) = \dfrac{47}{28}$ is a local minimum value.

(d) We have

$$f''(x) = (14x - 3)' = (14x)' - 3' = 14(x)' - 3' = 14 \cdot 1 - 0 = 14$$

Since $f'\left(\dfrac{3}{14}\right) = 0$ and $f''\left(\dfrac{3}{14}\right) = 14 > 0$, the Second Derivative Test tells us that f has a local minimum at $\dfrac{3}{14}$.

EXAMPLE: Let $f(x) = 5 - x - 9x^2$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

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(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

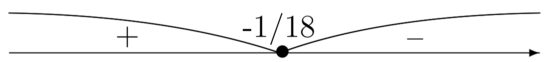
$$f'(x) = (5 - x - 9x^2)' = 5' - x' - (9x^2)' = 5' - x' - 9(x^2)' = 0 - 1 - 9 \cdot 2x = -1 - 18x$$

Since

$$-1 - 18x = 0 \iff -1 = 18x \iff x = -\frac{1}{18}$$

it follows that the critical number of f is $x = -\dfrac{1}{18}$.

(b) We have



Therefore f is increasing on $\left(-\infty, -\dfrac{1}{18}\right)$; it is decreasing on $\left(-\dfrac{1}{18}, \infty\right)$.

(c) Because $f'(x)$ changes from positive to negative at $-\dfrac{1}{18}$, the First Derivative Test tells us that $f\left(-\dfrac{1}{18}\right) = \dfrac{181}{36}$ is a local maximum value.

(d) We have

$$f''(x) = (-1 - 18x)' = (-1)' - (18x)' = (-1)' - 18(x)' = 0 - 18 \cdot 1 = -18$$

Since $f'\left(-\dfrac{1}{18}\right) = 0$ and $f''\left(-\dfrac{1}{18}\right) = -18 < 0$, the Second Derivative Test tells us that f has a local maximum at $-\dfrac{1}{18}$.

EXAMPLE: Let $f(x) = 2x^3 - 27x^2 + 4$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

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(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

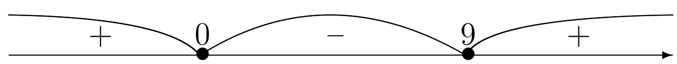
$$f'(x) = (2x^3-27x^2+4)' = (2x^3)'-(27x^2)'+4' = 2(x^3)'-27(x^2)'+4' = 2\cdot 3x^2-27\cdot 2x+0 = 6x^2-54x$$

Since

$$6x^2 - 54x = 0 \quad \Longleftrightarrow \quad 6x(x-9) = 0$$

it follows that the critical numbers of f are $x = 0$ and $x = 9$.

(b) We have



Therefore f is increasing on $(-\infty, 0)$ and $(9, \infty)$; it is decreasing on $(0, 9)$.

(c) Because $f'(x)$ changes from positive to negative at 0, the First Derivative Test tells us that $f(0) = 4$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 9, $f(9) = -725$ is a local minimum value.

(d) We have

$$f''(x) = (6x^2 - 54x)' = (6x^2)' - (54x)' = 6(x^2)' - 54(x)' = 6 \cdot 2x - 54 \cdot 1 = 12x - 54$$

Since $f'(0) = 0$ and $f''(0) = -54 < 0$, the Second Derivative Test tells us that f has a local maximum at 0. Similarly, since $f'(9) = 0$ and $f''(9) = 54 > 0$, f has a local minimum at 9.

EXAMPLE: Let $f(x) = 3x - 4x^3$.

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Solution:

(a) We have

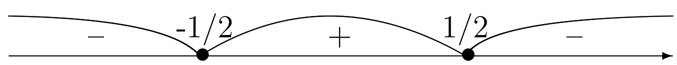
$$f'(x) = (3x - 4x^3)' = (3x)' - (4x^3)' = 3(x)' - 4(x^3)' = 3 \cdot 1 - 4 \cdot 3x^2 = 3 - 12x^2$$

Since

$$3 - 12x^2 = 0 \quad \Longleftrightarrow \quad 3 = 12x^2 \quad \Longleftrightarrow \quad x^2 = \frac{3}{12} = \frac{1}{4} \quad \Longleftrightarrow \quad x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

it follows that the critical numbers of f are $x = \dfrac{1}{2}$ and $x = -\dfrac{1}{2}$.

(b) We have



Therefore f is increasing on $(-1/2, 1/2)$; it is decreasing on $(-\infty, -1/2)$ and $(1/2, \infty)$.

(c) Because $f'(x)$ changes from positive to negative at $1/2$, the First Derivative Test tells us that $f\left(\dfrac{1}{2}\right) = 1$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at $-1/2$, $f\left(-\dfrac{1}{2}\right) = -1$ is a local minimum value.

(d) We have

$$f''(x) = (3 - 12x^2)' = 3' - (12x^2)' = 3' - 12(x^2)' = 0 - 12 \cdot 2x = -24x$$

Since $f'\left(\dfrac{1}{2}\right) = 0$ and $f''\left(\dfrac{1}{2}\right) = -12 < 0$, the Second Derivative Test tells us that f has a local maximum at $\dfrac{1}{2}$. Similarly, since $f'\left(-\dfrac{1}{2}\right) = 0$ and $f''\left(-\dfrac{1}{2}\right) = 12 > 0$, f has a local minimum at $-\dfrac{1}{2}$.

EXAMPLE: Let $f(x)=\dfrac{1}{3}x^3+x^2-15x+1$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:
(a) We have

$$\begin{aligned} f'(x)=\left(\frac{1}{3}x^3+x^2-15x+1\right)'=\left(\frac{1}{3}x^3\right)'+(x^2)'-(15x)'+1'&=\frac{1}{3}(x^3)'+(x^2)'-15(x)'+1'\\ &=\frac{1}{3}\cdot 3x^2+2x-15\cdot 1+0\\ &=x^2+2x-15\end{aligned}$$

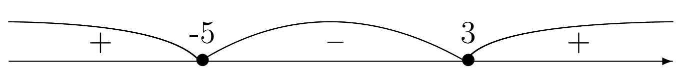
Since

$$x^2+2x-15=0 \quad\Longleftrightarrow\quad (x-3)(x+5)=0$$

it follows that the critical numbers of f are $x=3$ and $x=-5$.

(b) We have

Interval	$x-3$	$x+5$	$f'(x)$	f
$x<-5$	$-$	$-$	$+$	increasing on $(-\infty,-5)$
$-5<x<3$	$-$	$+$	$-$	decreasing on $(-5,3)$
$x>3$	$+$	$+$	$+$	increasing on $(3,\infty)$



Therefore f is increasing on $(-\infty,-5)$ and $(3,\infty)$; it is decreasing on $(-5,3)$.

(c) Because $f'(x)$ changes from positive to negative at -5, the First Derivative Test tells us that $f(-5)=\dfrac{178}{3}$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 3, $f(3)=26$ is a local minimum value.

(d) We have

$$f''(x)=(x^2+2x-15)'=(x^2)'+(2x)'-(15)'=(x^2)'+2(x)'-(15)'=2x+2\cdot 1-0=2x+2$$

Since $f'(-5)=0$ and $f''(-5)=-8<0$, the Second Derivative Test tells us that f has a local maximum at -5. Similarly, since $f'(3)=0$ and $f''(3)=8>0$, f has a local minimum at 3.

EXAMPLE: Let $f(x) = 3x^4 - 16x^3 + 24x^2 + 1$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Use the First Derivative Test to find the local extreme values of f.

(d) Use the Second Derivative Test to find the local extreme values of f.

Solution:

(a) We have

$$\begin{aligned} f'(x) = \left(3x^4 - 16x^3 + 24x^2 + 1\right)' &= \left(3x^4\right)' - \left(16x^3\right)' + \left(24x^2\right)' + 1' \\ &= 3\left(x^4\right)' - 16\left(x^3\right)' + 24\left(x^2\right)' + 1' \\ &= 3 \cdot 4x^3 - 16 \cdot 3x^2 + 24 \cdot 2x + 0 \\ &= 12x^3 - 48x^2 + 48x \end{aligned}$$

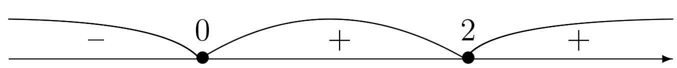
Since

$$12x^3 - 48x^2 + 48x = 0 \quad \Longleftrightarrow \quad 12x(x^2 - 4x + 4) = 0 \quad \Longleftrightarrow \quad 12x(x-2)^2 = 0$$

it follows that the critical numbers of f are $x = 0$ and $x = 2$.

(b) We have

Interval	$12x$	$(x-2)^2$	$f'(x)$	f
$x < 0$	$-$	$+$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$+$	$+$	increasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$



Therefore f is increasing on $(0, \infty)$; it is decreasing on $(-\infty, 0)$.

(c) Because $f'(x)$ changes from negative to positive at 0, the First Derivative Test tells us that $f(0) = 1$ is a local minimum value. Since $f'(x)$ does not change its sign at 2, $f(2) = 17$ is not a local extreme value.

(d) We have

$$\begin{aligned} f''(x) = (12x^3 - 48x^2 + 48x)' = (12x^3)' - (48x^2)' + (48x)' &= 12(x^3)' - 48(x^2)' + 48(x)' \\ &= 12 \cdot 3x^2 - 48 \cdot 2x + 48 \cdot 1 \\ &= 36x^2 - 96x + 48 \end{aligned}$$

Since $f'(0) = 0$ and $f''(0) = 48 > 0$, the Second Derivative Test tells us that f has a local maximum at 0. However, since $f'(2) = 0$ and $f''(2) = 0$, the Second Derivative Test test is inconclusive.

EXAMPLE: Let $f(x) = \dfrac{1}{x}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

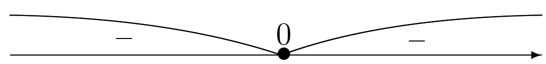
(c) Find the local extreme values of f, if any.

Solution:
(a) We have

$$f'(x) = \left(\frac{1}{x}\right)' = (x^{-1})' = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Note that there are no numbers at which f' is zero. The number at which f' does not exist is $x = 0$, but this number is not from the domain of f. Therefore f does not have critical numbers.

(b) We have



Therefore f is decreasing on $(-\infty, 0)$ and $(0, \infty)$.

(c) Since f does not have critical numbers, it does not have local extreme values.

EXAMPLE: Let $f(x) = \dfrac{3}{x^2}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

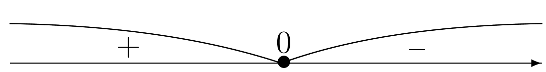
(c) Find the local extreme values of f, if any.

Solution:
(a) We have

$$f'(x) = \left(\frac{3}{x^2}\right)' = (3x^{-2})' = 3(x^{-2})' = 3 \cdot (-2)x^{-2-1} = -6x^{-3} = -\frac{6}{x^3}$$

Note that there are no numbers at which f' is zero. The number at which f' does not exist is $x = 0$, but this number is not from the domain of f. Therefore f does not have critical numbers.

(b) We have



Therefore f is increasing on $(-\infty, 0)$; it is decreasing on $(0, \infty)$.

(c) Since f does not have critical numbers, it does not have local extreme values.

EXAMPLE: Let $f(x) = x + \dfrac{4}{x}$.

(a) Find the critical numbers of f, if any.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f, if any.

Solution:

(a) We have

$$f'(x) = \left(x + \frac{4}{x}\right)' = (x + 4x^{-1})' = x' + (4x^{-1})' = x' + 4(x^{-1})'$$

$$= 1 + 4 \cdot (-1)x^{-1-1} = 1 - 4x^{-2} = 1 - \frac{4}{x^2}$$

Since

$$1 - \frac{4}{x^2} = 0 \iff 1 = \frac{4}{x^2} \iff x^2 = 4 \iff x = \pm 2$$

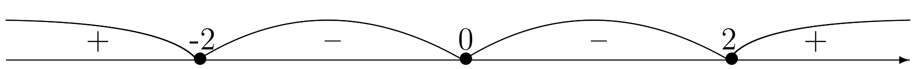
it follows that $f' = 0$ at $x = -2$ and $x = 2$. We also note that the number at which f' does not exist is $x = 0$, but this number is not from the domain of f. Therefore the critical numbers of f are $x = -2$ and $x = 2$ only.

(b) Note that

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2}{x^2} - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{x^2 - 2^2}{x^2} = \frac{(x-2)(x+2)}{x^2}$$

hence

Interval	x^2	$x-2$	$x+2$	$f'(x)$	f
$x < -2$	$+$	$-$	$-$	$+$	increasing on $(-\infty, -2)$
$-2 < x < 0$	$+$	$-$	$+$	$-$	decreasing on $(-2, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$



Therefore f is increasing on $(-\infty, -2)$ and $(2, \infty)$; it is decreasing on $(-2, 0)$ and $(0, 2)$.

(c) Because $f'(x)$ changes from positive to negative at -2, the First Derivative Test tells us that $f(-2) = -4$ is a local maximum value. Similarly, since $f'(x)$ changes from negative to positive at 2, $f(2) = 4$ is a local minimum value.

EXAMPLE: Let $f(x) = x \ln x - x$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

Solution:

(a) We have

$$f'(x) = (x \ln x - x)' = (x \ln x)' - x' = x' \ln x + x (\ln x)' - x'$$

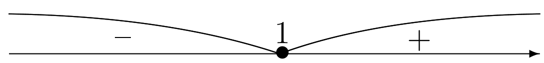
$$= 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 = \ln x + 1 - 1 = \ln x$$

Since

$$\ln x = 0 \iff x = 1$$

it follows that $f' = 0$ at $x = 1$. Therefore the critical number of f is $x = 1$.

(b) We have



Therefore f is decreasing on $(-\infty, 1)$; it is increasing on $(1, \infty)$.

(c) Because $f'(x)$ changes from negative to positive at 1, the First Derivative Test tells us that

$$f(1) = 1 \cdot \ln 1 - 1 = 1 \cdot 0 - 1 = 0 - 1 = -1$$

is a local minimum value.

EXAMPLE: Let $f(x) = 2x \ln x - 3x$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

EXAMPLE: Let $f(x)=2x\ln x-3x$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

Solution:

(a) We have

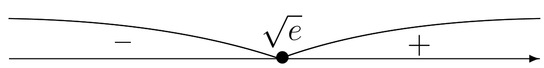
$$\begin{aligned}
f'(x)=(2x\ln x-3x)'=(2x\ln x)'-(3x)'&=2\,(x\ln x)'-3(x)'\\
&=2(x'\ln x+x\,(\ln x)')-3(x)'\\
&=2\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)-3\cdot 1\\
&=2\,(\ln x+1)-3=2\ln x+2-3=2\ln x-1
\end{aligned}$$

Since

$$2\ln x-1=0\quad\Longleftrightarrow\quad 2\ln x=1\quad\Longleftrightarrow\quad \ln x=\frac{1}{2}\quad\Longleftrightarrow\quad x=e^{1/2}=\sqrt{e}$$

it follows that $f'=0$ at $x=\sqrt{e}$. Therefore the critical number of f is $x=\sqrt{e}$.

(b) We have



Therefore f is decreasing on $(-\infty,\sqrt{e})$; it is increasing on $(\sqrt{e},\infty)$.

(c) Because $f'(x)$ changes from negative to positive at $\sqrt{e}$, the First Derivative Test tells us that

$$\begin{aligned}
f\left(\sqrt{e}\right)=2e^{1/2}\ln e^{1/2}-3e^{1/2}&=2e^{1/2}\cdot\frac{1}{2}\ln e-3e^{1/2}\\
&=2e^{1/2}\cdot\frac{1}{2}\cdot 1-3e^{1/2}=e^{1/2}-3e^{1/2}=-2e^{1/2}=-2\sqrt{e}
\end{aligned}$$

is a local minimum value.

EXAMPLE: Let $f(x)=(x-2)e^{-x}$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

EXAMPLE: Let $f(x)=(x-2)e^{-x}$.

(a) Find the critical numbers of f.

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the local extreme values of f.

Solution:
(a) We have

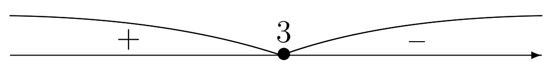
$$\begin{aligned} f'(x)=\Big((x-2)e^{-x}\Big)' &= (x-2)'e^{-x}+(x-2)\left(e^{-x}\right)' \\ &= (x'-2')e^{-x}+(x-2)e^{-x}\cdot(-x)' \\ &= (1-0)e^{-x}+(x-2)e^{-x}\cdot(-1) \\ &= 1\cdot e^{-x}-(x-2)e^{-x} \\ &= e^{-x}\Big(1-(x-2)\Big) \\ &= e^{-x}(1-x+2) \\ &= e^{-x}(3-x) \end{aligned}$$

Since

$$e^{-x}(3-x)=0 \quad\Longleftrightarrow\quad 3-x=0 \quad\Longleftrightarrow\quad x=3$$

it follows that $f'=0$ at $x=3$. Therefore the critical number of f is $x=3$.

(b) We have



Therefore f is increasing on $(-\infty,3)$; it is decreasing on $(3,\infty)$.

(c) Because $f'(x)$ changes from positive to negative at 3, the First Derivative Test tells us that

$$f(3)=(3-2)e^{-3}=1\cdot e^{-3}=e^{-3}$$

is a local maximum value.