The Mean Value Theorem

THEOREM (The Extreme Value Theorem): If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

THEOREM (Fermat's Theorem): If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

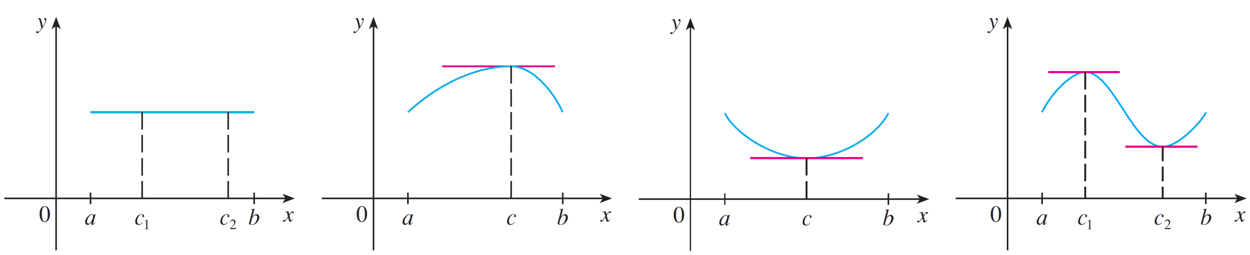
THEOREM (Rolle's Theorem): Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).

3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.



Proof: There are three cases:

Case I: $\boldsymbol{f(x) = k}$**, a constant.**
Then $f'(x) = 0$, so the number c can be taken to be *any* number in (a, b).

Case II: $\boldsymbol{f(x) > f(a)}$ **for some** $\boldsymbol{x}$ **in** $\boldsymbol{(a, b)}$**.**
By the Extreme Value Theorem, f has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a, b). Then f has a *local* maximum at c and, by hypothesis 2, f is differentiable at c. Therefore, $f'(c) = 0$ by Fermat's Theorem.

Case III: $\boldsymbol{f(x) < f(a)}$ **for some** $\boldsymbol{x}$ **in** $\boldsymbol{(a, b)}$**.**
By the Extreme Value Theorem, f has a minimum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this minimum value at a number c in the open interval (a, b). Then f has a *local* minimum at c and, by hypothesis 2, f is differentiable at c. Therefore, $f'(c) = 0$ by Fermat's Theorem.

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Solution: We first show that the equation $x^5+7x-2=0$ has a root. To this end we will use the Intermediate Value Theorem. In fact, let $f(x)=x^5+7x-2$. Then $f(0)=0^5+7\cdot 0-2=-2$ and $f(1)=1^5+7\cdot 1-2=6$. So, $f(0)$ is negative, $f(1)$ is positive. Since $f(x)$ is continuous, by the Intermediate Value Theorem there is a number c between 0 and 1 such that $f(c)=0$. Thus the given equation has a root.

Now we show that this root is unique, that is there are no other roots. We will prove it by a contradiction. Assume to the contrary that the equation $x^5+7x-2=0$ has at least two roots a and b, that is $f(a)=0$ and $f(b)=0$. Therefore $f(x)$ satisfies hypothesis 3 of Rolle's Theorem. Of course, f also satisfies hypotheses 1 and 2 of Rolle's Theorem since f is a polynomial. Thus by Rolle's Theorem there is a number c between a and b such that $f'(c)=0$. But this is impossible, since

$$f'(x)=5x^4+7>0$$

for any point x in $(-\infty,\infty)$. This gives a contradiction. Therefore, the equation can't have two roots.

EXAMPLE: Let f be a function continuous on $[0,1]$ and differentiable on $(0,1)$. Let also $f(0)=f(1)=0$. Prove that there exists a point c in $(0,1)$ such that $f'(c)=f(c)$.

Proof: Consider the following function

$$g(x)=f(x)e^{-x}$$

where x is in $[0,1]$. Since

$$g(0)=f(0)e^0=0\cdot 1=0 \quad \text{and} \quad g(1)=f(1)e^{-1}=0\cdot e^{-1}=0$$

it follows that $g(0)=g(1)$ and therefore g satisfies all three hypotheses of Rolle's Theorem (note that $g(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$ since both f and e^{-x} are continuous on $[0,1]$ and differentiable on $(0,1)$). By Rolle's Theorem we have $g'(c)=0$ at some point c in $(0,1)$. But

$$\begin{aligned} g'(x)=(f(x)e^{-x})' &= f'(x)e^{-x}+f(x)(e^{-x})' \\ &= f'(x)e^{-x}-f(x)e^{-x} \\ &= e^{-x}(f'(x)-f(x)) \end{aligned}$$

hence

$$0=g'(c)=e^{-c}(f'(c)-f(c))$$

therefore

$$0=f'(c)-f(c)$$

which gives

$$f'(c)=f(c)$$

THE MEAN VALUE THEOREM: Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a,b]$.

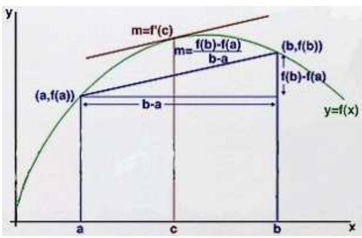
2. f is differentiable on the open interval (a,b).

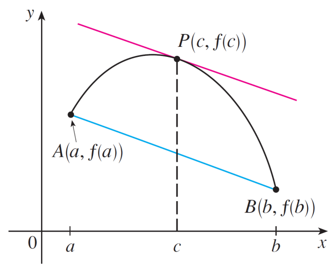
Then there is a number c in (a,b) such that

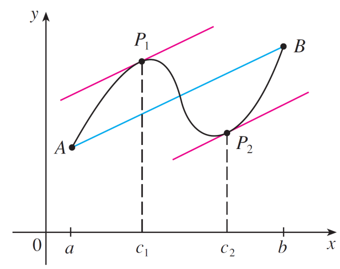
$$\boxed{f'(c)=\frac{f(b)-f(a)}{b-a}}$$

or, equivalently,

$$\boxed{f(b)-f(a)=f'(c)(b-a)}$$







Proof: Consider the following function

$$h(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$$

Note that this function satisfies all three hypotheses of Rolle's Theorem. In fact, the function h is continuous on $[a,b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous. The function h is differentiable on (a,b) because both f and the first-degree polynomial are differentiable. Now we show that $h(a)=h(b)$. In fact,

$$h(a)=\underbrace{f(a)-f(a)}_{=0}-\frac{f(b)-f(a)}{b-a}(\underbrace{a-a}_{=0})=0$$

and

$$h(b)=f(b)-f(a)-\frac{f(b)-f(a)}{b-a}(b-a)=f(b)-f(a)-[f(b)-f(a)]=0$$

So, $h(a)=0,\ h(b)=0$, therefore $h(a)=h(b)$.

In follows that h satisfies all three hypotheses of Rolle's Theorem by which there is a number c in (a,b) such that $h'(c)=0$. Note that

$$h'(x)=\left(f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)\right)'=f'(x)-(f(a))'-\left(\frac{f(b)-f(a)}{b-a}(x-a)\right)'$$

$$=f'(x)-(f(a))'-\frac{f(b)-f(a)}{b-a}(x-a)'=f'(x)-0-\frac{f(b)-f(a)}{b-a}\cdot 1=f'(x)-\frac{f(b)-f(a)}{b-a}$$

hence

$$0=h'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}$$

So, $0=f'(c)-\dfrac{f(b)-f(a)}{b-a}$ which gives the desired result.

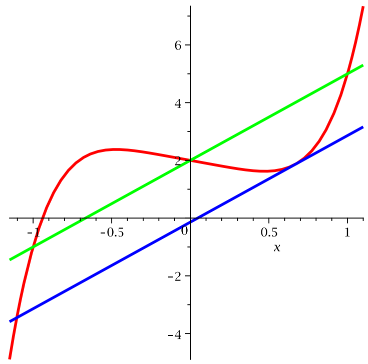
EXAMPLE: Let $f(x)=4x^5-x+2,\ a=0,\ b=1$. Since f is a polynomial, it satisfies first two hypotheses of the Mean-Value Theorem. Therefore there is a number c in $(0,1)$ such that

$$f(1)-f(0)=f'(c)(1-0)$$

Now $f(1)=5,\ f(0)=2$ and $f'(x)=20x^4-1$, so this equation becomes

$$5-2=20c^4-1 \quad\Longrightarrow\quad 4=20c^4 \quad\Longrightarrow\quad \frac{4}{20}=c^4 \quad\Longrightarrow\quad c=\pm\sqrt[4]{\frac{4}{20}}=\pm\frac{1}{\sqrt[4]{5}}$$

But c must lie in $(0,1)$, so $c=\dfrac{1}{\sqrt[4]{5}}$.



EXAMPLE: Suppose that $f(0)=-2$ and $f'(x)\le 8$ for all values of x. How large can $f(1)$ possibly be?

Solution: We are given that f is differentiable everywhere, therefore it is continuous everywhere. Thus, we can use the Mean Value Theorem by which there exists a number c in $[0,1]$ such that

$$f(1)-f(0)=f'(c)(1-0) \quad\Longrightarrow\quad f(1)=f(0)+f'(c)(1-0)=-2+f'(c)\le -2+8=6$$

So, the largest possible value for $f(1)$ is ≤ 6. Finally, if $f(x)=8x-2$, then $f(1)=6$, therefore the largest possible value for $f(1)$ is 6.

THEOREM: If $f'(x)=0$ for all x in an interval (a,b), then f is constant on (a,b).

Proof: Let x_1 and x_2 be any two numbers in (a,b) with $x_1<x_2$. Since f is differentiable on (a,b), it must be differentiable on (x_1,x_2) and continuous on $[x_1,x_2]$. Therefore we can apply the Mean Value Theorem to f on the interval $[x_1,x_2]$ by which there exists a number c in (x_1,x_2) such that

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)$$

Since $f'(x)=0$ for all x, we have

$$f(x_2)-f(x_1)=0\cdot(x_2-x_1)=0 \quad\Longrightarrow\quad f(x_2)=f(x_1)$$

Therefore, f has the same value at *any* two numbers x_1 and x_2 in (a,b). This means that f is constant on (a,b).

COROLLARY: If $f'(x)=g'(x)$ for all x in an interval (a,b), then $f-g$ is constant on (a,b); that is, $f(x)=g(x)+c$ where c is a constant.

COROLLARY: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

Proof: Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all x in (a, b). Thus, by the Theorem above, F is constant; that is, $f - g$ is constant.

REMARK: Care must be taken in applying the above Theorem. In fact, let

$$f(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all x in D. But f is obviously not a constant function. This does not contradict the above Theorem because D is not an interval.