The Mean Value Theorem

THEOREM (The Extreme Value Theorem): If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.

THEOREM (Fermat’s Theorem): If $f$ has a local maximum or minimum at $c$, and if $f'(c)$ exists, then $f'(c) = 0$.

THEOREM (Rolle’s Theorem): Let $f$ be a function that satisfies the following three hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.
3. $f(a) = f(b)$

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$.

Proof: There are three cases:

Case I: $f(x) = k$, a constant.
Then $f'(x) = 0$, so the number $c$ can be taken to be any number in $(a, b)$.

Case II: $f(x) > f(a)$ for some $x$ in $(a, b)$.
By the Extreme Value Theorem, $f$ has a maximum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number $c$ in the open interval $(a, b)$. Then $f$ has a local maximum at $c$ and, by hypothesis 2, $f$ is differentiable at $c$. Therefore, $f'(c) = 0$ by Fermat’s Theorem.

Case III: $f(x) < f(a)$ for some $x$ in $(a, b)$.
By the Extreme Value Theorem, $f$ has a minimum value somewhere in $[a, b]$. Since $f(a) = f(b)$, it must attain this minimum value at a number $c$ in the open interval $(a, b)$. Then $f$ has a local minimum at $c$ and, by hypothesis 2, $f$ is differentiable at $c$. Therefore, $f'(c) = 0$ by Fermat’s Theorem.

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Solution: We first show that the equation \( x^5 + 7x - 2 = 0 \) has a root. To this end we will use the Intermediate Value Theorem. In fact, let \( f(x) = x^5 + 7x - 2 \). Then \( f(0) = 0^5 + 7 \cdot 0 - 2 = -2 \) and \( f(1) = 1^5 + 7 \cdot 1 - 2 = 6 \). So, \( f(0) \) is negative, \( f(1) \) is positive. Since \( f(x) \) is continuous, by the Intermediate Value Theorem there is a number \( c \) between 0 and 1 such that \( f(c) = 0 \). Thus the given equation has a root.

Now we show that this root is unique, that is there are no other roots. We will prove it by a contradiction. Assume to the contrary that the equation \( x^5 + 7x - 2 = 0 \) has at least two roots \( a \) and \( b \), that is \( f(a) = 0 \) and \( f(b) = 0 \). Therefore \( f(x) \) satisfies hypothesis 3 of Rolle’s Theorem. Of course, \( f(x) \) also satisfies hypotheses 1 and 2 of Rolle’s Theorem since \( f(x) \) is a polynomial. Thus by Rolle’s Theorem there is a number \( c \) between \( a \) and \( b \) such that \( f'(c) = 0 \). But this is impossible, since

\[
f'(x) = 5x^4 + 7 > 0
\]

for any point \( x \) in \((-\infty, \infty)\). This gives a contradiction. Therefore, the equation can’t have two roots.

EXAMPLE: Let \( f \) be a function continuous on \([0,1]\) and differentiable on \((0,1)\). Let also \( f(0) = f(1) = 0 \). Prove that there exists a point \( c \) in \((0,1)\) such that \( f'(c) = f(c) \).

Proof: Consider the following function

\[
g(x) = f(x)e^{-x}
\]

where \( x \) is in \([0,1]\). Since

\[
g(0) = f(0)e^0 = 0 \cdot 1 = 0 \quad \text{and} \quad g(1) = f(1)e^{-1} = 0 \cdot e^{-1} = 0
\]

it follows that \( g(0) = g(1) \) and therefore \( g \) satisfies all three hypotheses of Rolle’s Theorem (note that \( g(x) \) is continuous on \([0,1]\) and differentiable on \((0,1)\) since both \( f \) and \( e^{-x} \) are continuous on \([0,1]\) and differentiable on \((0,1)\)). By Rolle’s Theorem we have \( g'(c) = 0 \) at some point \( c \) in \((0,1)\). But

\[
g'(x) = (f(x)e^{-x})' = f'(x)e^{-x} + f(x)(e^{-x})' = f'(x)e^{-x} - f(x)e^{-x} = e^{-x}(f'(x) - f(x))
\]

hence

\[
0 = g'(c) = e^{-c}(f'(c) - f(c))
\]

therefore

\[
0 = f'(c) - f(c)
\]

which gives

\[
f'(c) = f(c)
\]
THE MEAN VALUE THEOREM: Let \( f \) be a function that satisfies the following hypotheses:

1. \( f \) is continuous on the closed interval \([a, b]\).
2. \( f \) is differentiable on the open interval \((a, b)\).

Then there is a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

or, equivalently,

\[
f(b) - f(a) = f'(c)(b - a)
\]

Proof: Consider the following function

\[
h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)
\]

Note that this function satisfies all three hypotheses of Rolle’s Theorem. In fact, the function \( h \) is continuous on \([a, b]\) because it is the sum of \( f \) and a first-degree polynomial, both of which are continuous. The function \( h \) is differentiable on \((a, b)\) because both \( f \) and the first-degree polynomial are differentiable. Now we show that \( h(a) = h(b) \).

In follows that \( h \) satisfies all three hypotheses of Rolle’s Theorem by which there is a number \( c \) in \((a, b)\) such that \( h'(c) = 0 \). Note that

\[
h'(x) = \left( f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \right)' = f'(x) - (f(a))' - \left( \frac{f(b) - f(a)}{b - a} \right)'(x - a)
\]

\[
= f'(x) - (f(a))' - \frac{f(b) - f(a)}{b - a}(x - a)' = f'(x) - 0 - \frac{f(b) - f(a)}{b - a} \cdot 1 = f'(x) - \frac{f(b) - f(a)}{b - a}
\]

hence

\[
0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}
\]

So, \( 0 = f'(c) - \frac{f(b) - f(a)}{b - a} \) which gives the desired result.
EXAMPLE: Let \( f(x) = 4x^5 - x + 2 \), \( a = 0 \), \( b = 1 \). Since \( f \) is a polynomial, it satisfies first two hypotheses of the Mean-Value Theorem. Therefore there is a number \( c \) in \((0, 1)\) such that

\[
f(1) - f(0) = f'(c)(1 - 0)
\]

Now \( f(1) = 5 \), \( f(0) = 2 \) and \( f'(x) = 20x^4 - 1 \), so this equation becomes

\[
5 - 2 = 20c^4 - 1 \implies 4 = 20c^4 \implies c = \pm \sqrt{\frac{4}{20}} = \pm \frac{1}{\sqrt{5}}
\]

But \( c \) must lie in \((0, 1)\), so \( c = \frac{1}{\sqrt{5}} \).

EXAMPLE: Suppose that \( f(0) = -2 \) and \( f'(x) \leq 8 \) for all values of \( x \). How large can \( f(1) \) possibly be?

Solution: We are given that \( f \) is differentiable everywhere, therefore it is continuous everywhere. Thus, we can use the Mean Value Theorem by which there exists a number \( c \) in \([0, 1]\) such that

\[
f(1) - f(0) = f'(c)(1 - 0) \implies f(1) = f(0) + f'(c)(1 - 0) = -2 + f'(c) \leq -2 + 8 = 6
\]

So, the largest possible value for \( f(1) \) is \( \leq 6 \). Finally, if \( f(x) = 8x - 2 \), then \( f(1) = 6 \), therefore the largest possible value for \( f(1) \) is 6.

THEOREM: If \( f'(x) = 0 \) for all \( x \) in an interval \((a, b)\), then \( f \) is constant on \((a, b)\).

Proof: Let \( x_1 \) and \( x_2 \) be any two numbers in \((a, b)\) with \( x_1 < x_2 \). Since \( f \) is differentiable on \((a, b)\), it must be differentiable on \((x_1, x_2)\) and continuous on \([x_1, x_2]\). Therefore we can apply the Mean Value Theorem to \( f \) on the interval \([x_1, x_2]\) by which there exists a number \( c \) in \((x_1, x_2)\) such that

\[
f(x_2) - f(x_1) = f'(c)(x_2 - x_1)
\]

Since \( f'(x) = 0 \) for all \( x \), we have

\[
f(x_2) - f(x_1) = 0 \cdot (x_2 - x_1) = 0 \implies f(x_2) = f(x_1)
\]

Therefore, \( f \) has the same value at any two numbers \( x_1 \) and \( x_2 \) in \((a, b)\). This means that \( f \) is constant on \((a, b)\).

COROLLARY: If \( f'(x) = g'(x) \) for all \( x \) in an interval \((a, b)\), then \( f - g \) is constant on \((a, b)\); that is, \( f(x) = g(x) + c \) where \( c \) is a constant.
COROLLARY: If $f'(x) = g'(x)$ for all $x$ in an interval $(a, b)$, then $f - g$ is constant on $(a, b)$; that is, $f(x) = g(x) + c$ where $c$ is a constant.

Proof: Let $F(x) = f(x) - g(x)$. Then

$$F'(x) = f'(x) - g'(x) = 0$$

for all $x$ in $(a, b)$. Thus, by the Theorem above, $F$ is constant; that is, $f - g$ is constant.

REMARK: Care must be taken in applying the above Theorem. In fact, let

$$f(x) = \frac{x}{|x|} = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0
\end{cases}$$

The domain of $f$ is $D = \{x \mid x \neq 0\}$ and $f'(x) = 0$ for all $x$ in $D$. But $f$ is obviously not a constant function. This does not contradict the above Theorem because $D$ is not an interval.