Maximum and Minimum Values

DEFINITION: A function \( f \) has an **absolute maximum** (or **global maximum**) at \( c \) if \( f(c) \geq f(x) \) for all \( x \) in \( D \), where \( D \) is the domain of \( f \). The number \( f(c) \) is called the **maximum value** of \( f \) on \( D \). Similarly, \( f \) has an **absolute minimum** (or **global minimum**) at \( c \) if \( f(c) \leq f(x) \) for all \( x \) in \( D \) and the number \( f(c) \) is called the **minimum value** of \( f \) on \( D \). The maximum and minimum values of \( f \) are called the **extreme values** of \( f \).

DEFINITION: A function \( f \) has a **local maximum** (or **relative maximum**) at \( c \) if \( f(c) \geq f(x) \) when \( x \) is near \( c \). [This means that \( f(c) \geq f(x) \) for all \( x \) in some open interval containing \( c \).] Similarly, \( f \) has a **local minimum** (or **relative minimum**) at \( c \) if \( f(c) \leq f(x) \) when \( x \) is near \( c \).

**EXAMPLES:**

1. The function \( f(x) = x^2 \) has the absolute (and local) minimum at \( x = 0 \) and has no absolute or local maximum.

2. The function \( f(x) = x^2, \ x \in (-\infty, 0) \cup (0, \infty) \), has no absolute or local minimum and no absolute or local maximum.

3. The functions \( f(x) = x, \ x^3, \ x^5, \) etc. have no absolute or local minima and no absolute or local maxima.
4. The functions $f(x) = \sin x, \cos x$ have infinitely many absolute and local minima and maxima.

5. The functions $f(x) = \sec x, \csc x$ have infinitely many local minima and maxima and no absolute minima and maxima.

6. The functions $f(x) = \tan x, \cot x$ have no absolute or local minima and no absolute or local maxima.
7. The function \( f(x) = x^4 + x^3 - 11x^2 - 9x + 18 = (x - 3)(x - 1)(x + 2)(x + 3) \) has the absolute minimum at \( x \approx 2.2 \) and has no absolute maximum. It has two local minima at \( x \approx -2.6 \) and \( x \approx 2.2 \) and the local maximum at \( x \approx -0.4 \).

8. The function \( f(x) = x, \ 1 \leq x \leq 2 \), has the absolute minimum at \( x = 1 \) and absolute maximum at \( x = 2 \). It has no local maximum or minimum.

9. The function \( f(x) = x, \ 1 < x < 2 \), has no absolute or local minimum and no absolute or local maximum.

10. The function \( f(x) = x \) has no absolute or local minimum and no absolute or local maximum.

11. The function \( f(x) = 1 \) has the absolute (and local) minimum and absolute (and local) maximum at any point on the number line.
THEOREM (The Extreme Value Theorem): If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.

THEOREM (Fermat’s Theorem): If $f$ has a local maximum or minimum at $c$, and if $f'(c)$ exists, then $f'(c) = 0$.

REMARK 1: The converse of this theorem is not true. In other words, when $f'(c) = 0$, $f$ does not necessarily have a local maximum or minimum. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ equals 0 at $x = 0$, but $x = 0$ is not a point of a local minimum or maximum.

REMARK 2: Sometimes $f'(c)$ does not exist, but $x = c$ is a point of a local maximum or minimum. For example, if $f(x) = |x|$, then $f'(0)$ does not exist. But $f(x)$ has its local (and absolute) minimum at $x = 0$. 
**DEFINITION:** A **critical number** of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

**REMARK:** From Fermat’s Theorem it follows that if $f$ has a local maximum or minimum at $c$, then $c$ is a critical number of $f$.

**EXAMPLES:**

(a) If $f(x) = 2x^2 + 5x - 1$, then $f'(x) = 4x + 5$. Hence the only critical number of $f$ is $x = -\frac{5}{4}$.

(b) If $f(x) = \sqrt[3]{x^2}$, then $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$. Hence the only critical number of $f$ is $x = 0$.

(c) If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$. Since $x = 0$ is not in the domain, $f$ has no critical numbers.

**EXAMPLES:**

(a) Find the critical numbers of $f(x) = 2x^3 - 9x^2 + 12x - 5$.

(b) Find the critical numbers of $f(x) = 2x + 3\sqrt[3]{x^2}$. 
EXAMPLES:

(a) Find the critical numbers of \( f(x) = 2x^3 - 9x^2 + 12x - 5 \).

Solution: We have
\[
f'(x) = 6x^2 - 18x + 12
= 6(x^2 - 3x + 2)
= 6(x - 1)(x - 2)
\]
thus \( f'(x) = 0 \) at \( x = 1 \) and \( x = 2 \). Since \( f'(x) \) exists everywhere, \( x = 1 \) and \( x = 2 \) are the only critical numbers.

(b) Find the critical numbers of \( f(x) = 2x + 3\sqrt[3]{x^2} \).

Solution: We have
\[
f'(x) = (2x + 3x^{2/3})'
= 2x' + 3(x^{2/3})'
= 2 + 3 \cdot \frac{2}{3} x^{-1/3}
= 2 + 2x^{-1/3}
= \left\{ 2x^{-1/3} \cdot x^{1/3} + 2x^{-1/3} \cdot 1 = 2x^{-1/3}(x^{1/3} + 1) \right\} = \frac{2(\sqrt[3]{x} + 1)}{\sqrt[3]{x}}
\]
In short,
\[
f'(x) = 2 + 2x^{-1/3} = 2x^{-1/3}(x^{1/3} + 1) = \frac{2(\sqrt[3]{x} + 1)}{\sqrt[3]{x}}
\]
It follows that \( f'(x) \) equals 0 when \( \sqrt[3]{x} + 1 = 0 \), which is at \( x = -1 \); \( f'(x) \) does not exist when \( \sqrt[3]{x} = 0 \), which is at \( x = 0 \). Thus the critical numbers are \( -1 \) and \( 0 \).

EXAMPLE: Find the critical numbers of \( f(x) = x^{1/3}(2 - x) \).
EXAMPLE: Find the critical numbers of \( f(x) = x^{1/3}(2 - x) \).

Solution: We have

\[
f'(x) = \left[ x^{1/3}(2 - x) \right]' \]
\[
= (x^{1/3})'(2 - x) + x^{1/3}(2 - x)' \]
\[
= \frac{1}{3} x^{-2/3}(2 - x) + x^{1/3} \cdot (-1) \]
\[
= \frac{2 - x}{3x^{2/3}} - x^{1/3} \]
\[
= \left\{ \frac{2 - x}{3x^{2/3}} - \frac{x^{1/3} \cdot 3x^{2/3}}{3x^{2/3}} \right\} = 2 \left( \frac{1 - 2x}{3x^{2/3}} \right)
\]

In short,

\[
f'(x) = (x^{1/3})'(2 - x) + x^{1/3}(2 - x)' = \frac{2 - x}{3x^{2/3}} - x^{1/3} = \frac{2(1 - 2x)}{3x^{2/3}}
\]

Here is another way to get the same result:

\[
f'(x) = \left[ x^{1/3}(2 - x) \right]' \]
\[
= (2x^{1/3} - x^{4/3})' \]
\[
= 2(x^{1/3})' - (x^{4/3})' \]
\[
= 2 \cdot \frac{1}{3} x^{1/3-1} - 4 \cdot x^{4/3-1} \]
\[
= \frac{2}{3} x^{-2/3} - \frac{4}{3} x^{1/3} = \left\{ \frac{2}{3} x^{-2/3} - \frac{4}{3} x^{1/3} \cdot \frac{x^{2/3}}{x^{2/3}} \right\} = \frac{2}{3} x^{-2/3} - \frac{4}{3} x^{1/3} \cdot x^{2/3} = \frac{2}{3} x^{-2/3} - \frac{4}{3} x^{1/3} \cdot x^{2/3} = \frac{2 - 4x}{3x^{2/3}} \]
\[
= \frac{2(1 - 2x)}{3x^{2/3}}
\]

or

\[
= \frac{2}{3} x^{-2/3} - \frac{4}{3} x^{1/3} = \left\{ \frac{2}{3} x^{-2/3} \cdot 1 - \frac{2}{3} x^{-2/3} \cdot 2x = \frac{2}{3} x^{-2/3}(1 - 2x) \right\} = \frac{2(1 - 2x)}{3x^{2/3}}
\]

In short,

\[
f'(x) = (2x^{1/3} - x^{4/3})' = \frac{2}{3} x^{-2/3} - \frac{4}{3} x^{1/3} = \frac{2}{3} x^{-2/3}(1 - 2x) = \frac{2(1 - 2x)}{3x^{2/3}}.
\]

It follows that \( f'(x) \) equals 0 when \( 1 - 2x = 0 \), which is at \( x = \frac{1}{2} \); \( f'(x) \) does not exist when \( x^{2/3} = 0 \), which is at \( x = 0 \). Thus the critical numbers are \( \frac{1}{2} \) and 0.
THE CLOSED INTERVAL METHOD: To find the absolute maximum and minimum values of a continuous function $f$ on a closed interval $[a,b]$:

1. Find the values of $f$ at the critical numbers of $f$ in $(a,b)$.
2. Find the values of $f$ at the endpoints of the interval.
3. The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest value of these values is the absolute minimum value.

EXAMPLE: Find the absolute maximum and minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1,5]$ and determine where these values occur.

Solution:

**Step 1:** Since

$$f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x - 2)(x - 3)$$

there are two critical numbers $x = 2$ and $x = 3$.

**Step 2:** We now evaluate $f$ at these critical numbers and at the endpoints $x = 1$ and $x = 5$. We have

$$f(1) = 23, \quad f(2) = 28, \quad f(3) = 27, \quad f(5) = 55$$

**Step 3:** The largest value is 55 and the smallest value is 23. Therefore the absolute maximum of $f$ on $[1,5]$ is 55, occurring at $x = 5$ and the absolute minimum of $f$ on $[1,5]$ is 23, occurring at $x = 1$.

EXAMPLES:

(a) Find the absolute maximum and minimum values of $f(x) = 2x^3 - 15x^2 + 24x + 2$ on $[0,2]$ and determine where these values occur.

(b) Find the absolute maximum and minimum values of $f(x) = 6x^{4/3} - 3x^{1/3}$ on the interval $[-1,1]$ and determine where these values occur.
EXAMPLES:

(a) Find the absolute maximum and minimum values of \( f(x) = 2x^3 - 15x^2 + 24x + 2 \) on \([0, 2]\) and determine where these values occur.

Solution:

**Step 1:** Since
\[
f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x - 4)(x - 1)
\]
there are two critical numbers \( x = 1 \) and \( x = 4 \).

**Step 2:** Since \( x = 4 \) is not from \([0, 2]\), we evaluate \( f \) only at \( x = 1 \) and at the endpoints \( x = 0 \) and \( x = 2 \). We have
\[
f(0) = 2, \quad f(1) = 13, \quad f(2) = 6
\]

**Step 3:** The largest value is 13 and the smallest value is 2. Therefore the absolute maximum of \( f \) on \([0, 2]\) is 13, occurring at \( x = 1 \) and the absolute minimum of \( f \) on \([0, 2]\) is 2, occurring at \( x = 0 \).

(b) Find the absolute maximum and minimum values of \( f(x) = 6x^{4/3} - 3x^{1/3} \) on the interval \([-1, 1]\) and determine where these values occur.

Solution:

**Step 1:** Since
\[
f'(x) = (6x^{4/3} - 3x^{1/3})' = 6(x^{4/3})' - 3(x^{1/3})' = 6 \cdot \frac{4}{3}x^{1/3} - 3 \cdot \frac{1}{3}x^{-2/3}
\]
\[
= 8x^{1/3} - x^{-2/3}
\]
\[
= 8x^{-2/3} \cdot x - 1 \cdot x^{-2/3}
\]
\[
= x^{-2/3}(8x - 1)
\]
\[
= \frac{8x - 1}{x^{2/3}}
\]
there are two critical numbers \( x = 0 \) and \( x = \frac{1}{8} \).

**Step 2:** We now evaluate \( f \) at these critical numbers and at the endpoints \( x = -1 \) and \( x = 1 \). We have
\[
f(-1) = 9, \quad f(0) = 0, \quad f\left(\frac{1}{8}\right) = -\frac{9}{8}, \quad f(1) = 3
\]

**Step 3:** The largest value is 9 and the smallest value is \(-\frac{9}{8}\). Therefore the absolute maximum of \( f \) on \([-1, 1]\) is 9, occurring at \( x = -1 \) and the absolute minimum of \( f \) on \([-1, 1]\) is \(-\frac{9}{8}\), occurring at \( x = \frac{1}{8} \).
Appendix

EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = 2x + 3 \) on \([-1, 4]\) and determine where these values occur.

Solution:

**Step 1:** Since

\[
 f'(x) = (2x + 3)' = (2x)' - 3' = 2(x)' - 3' = 2 \cdot 1 - 0 = 2
\]

there are no critical numbers.

**Step 2:** Since there are no critical numbers, we evaluate \( f \) at the endpoints \( x = -1 \) and \( x = 4 \) only. We have

\[
 f(-1) = 2(-1) + 3 = -2 + 3 = 1 \\
 f(4) = 2(4) + 3 = 8 + 3 = 11
\]

**Step 3:** The largest value is 11 and the smallest value is 1. Therefore the absolute maximum of \( f \) on \([-1, 4]\) is 11, occurring at \( x = 4 \) and the absolute minimum of \( f \) on \([-1, 4]\) is 1, occurring at \( x = -1 \).

EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = 2x^2 - 8x + 1 \) on \([-1, 3]\) and determine where these values occur.

Solution:

**Step 1:** Since

\[
 f'(x) = (2x^2 - 8x + 1)' = (2x^2)' - (8x)' + 1' = 2(x^2)' - 8(x)' + 1' = 2 \cdot 2x - 8 \cdot 1 + 0 = 4x - 8
\]

the number at which \( f' \) is zero is 2. Therefore \( x = 2 \) is the critical number.

**Step 2:** We evaluate \( f \) at the critical number \( x = 2 \) and at the endpoints \( x = -1, 3 \). We have

\[
 f(2) = 2 \cdot 2^2 - 8 \cdot 2 + 1 = 2 \cdot 4 - 8 \cdot 2 + 1 = 8 - 16 + 1 = -7 \\
 f(-1) = 2(-1)^2 - 8(-1) + 1 = 2 + 8 + 1 = 11 \\
 f(3) = 2 \cdot 3^2 - 8 \cdot 3 + 1 = 2 \cdot 9 - 8 \cdot 3 + 1 = 18 - 24 + 1 = -5
\]

**Step 3:** The largest value is 11 and the smallest value is \(-7\). Therefore the absolute maximum of \( f \) on \([-1, 3]\) is 11, occurring at \( x = -1 \) and the absolute minimum of \( f \) on \([-1, 3]\) is \(-7\), occurring at \( x = 2 \).

EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = 5 - x^3 \) on \([-2, 1]\) and determine where these values occur.
EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = 5 - x^3 \) on \([-2, 1]\) and determine where these values occur.

Solution:

**Step 1:** Since
\[
f'(x) = (5 - x^3)' = 5' - (x^3)' = 0 - 3x^2 = -3x^2
\]
the critical number is \( x = 0 \).

**Step 2:** We evaluate \( f \) at the critical number \( x = 0 \) and at the endpoints \( x = -2, 1 \). We have
\[
\begin{align*}
  f(0) &= 5 - 0^3 = 5 - 0 = 5 \\
  f(-2) &= 5 - (-2)^3 = 5 - (-8) = 5 + 8 = 13 \\
  f(1) &= 5 - 1^3 = 5 - 1 = 4
\end{align*}
\]

**Step 3:** The largest value is 13 and the smallest value is 4. Therefore the absolute maximum of \( f \) on \([-2, 1]\) is 13, occurring at \( x = -2 \) and the absolute minimum of \( f \) on \([-2, 1]\) is 4, occurring at \( x = 1 \).

EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = x^3 + 3x^2 - 1 \) on \([-3, 2]\) and determine where these values occur.

Solution:

**Step 1:** Since
\[
f'(x) = (x^3 + 3x^2 - 1)' = (x^3)' + (3x^2)' - 1' = (x^3)' + 3(x^2)' - 1' = 3x^2 + 3 \cdot 2x - 0 = 3x^2 + 6x = 3x(x + 2)
\]
the critical numbers are \( x = 0 \) and \( x = -2 \).

**Step 2:** We evaluate \( f \) at the critical numbers \( x = 0, -2 \) and at the endpoints \( x = -3, 2 \). We have
\[
\begin{align*}
  f(0) &= 0^3 + 3 \cdot 0^2 - 1 = 0 + 0 - 1 = -1 \\
  f(-2) &= (-2)^3 + 3 \cdot (-2)^2 - 1 = -8 + 12 - 1 = 3 \\
  f(-3) &= (-3)^3 + 3 \cdot (-3)^2 - 1 = -27 + 27 - 1 = -1 \\
  f(2) &= 2^3 + 3 \cdot 2^2 - 1 = 8 + 12 - 1 = 19
\end{align*}
\]

**Step 3:** The largest value is 19 and the smallest value is -1. Therefore the absolute maximum of \( f \) on \([-3, 2]\) is 19, occurring at \( x = 2 \) and the absolute minimum of \( f \) on \([-3, 2]\) is -1, occurring at \( x = 0 \) and \( x = -3 \).
EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = 3x^4 + 4x^3 - 36x^2 \) on \([-2, 3]\) and determine where these values occur.

Solution:

**Step 1:** Since

\[
f'(x) = (3x^4 + 4x^3 - 36x^2)' = (3x^4)' + (4x^3)' - (36x^2)' = 3(x^4)' + 4(x^3)' - 36(x^2)'
\]

\[
= 3 \cdot 4x^3 + 4 \cdot 3x^2 - 36 \cdot 2x
\]

\[
= 12x^3 + 12x^2 - 72x
\]

\[
= 12x(x^2 + x - 6)
\]

\[
= 12x(x - 2)(x + 3)
\]

the critical numbers are \( x = 0, \ x = 2, \) and \( x = -3. \)

**Step 2:** Since \(-3 \notin [-2, 3]\), we evaluate \( f \) only at the critical numbers \( x = 0, 2 \) and at the endpoints \( x = -2, 3. \) We have

\[
f(0) = 3 \cdot 0^4 + 4 \cdot 0^3 - 36 \cdot 0^2 = 0 + 0 - 0 = 0
\]

\[
f(2) = 3 \cdot 2^4 + 4 \cdot 2^3 - 36 \cdot 2^2 = 48 + 32 - 144 = -64
\]

\[
f(-2) = 3 \cdot (-2)^4 + 4 \cdot (-2)^3 - 36 \cdot (-2)^2 = 48 - 32 - 144 = -128
\]

\[
f(3) = 3 \cdot 3^4 + 4 \cdot 3^3 - 36 \cdot 3^2 = 243 + 108 - 324 = 27
\]

**Step 3:** The largest value is 27 and the smallest value is -128. Therefore the absolute maximum of \( f \) on \([-2, 3]\) is 27, occurring at \( x = 3 \) and the absolute minimum of \( f \) on \([-2, 3]\) is -128, occurring at \( x = -2. \)

EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = -\frac{1}{x^2} \) on \([0.4, 5]\) and determine where these values occur.

Solution:

**Step 1:** We have

\[
f'(x) = \left( -\frac{1}{x^2} \right)' = - \left( \frac{1}{x^2} \right)' = - \left( x^{-2} \right)' = -(-2)x^{-2-1} = 2x^{-3} = \frac{2}{x^3}
\]

Note that there are no numbers at which \( f' \) is zero. The number at which \( f' \) does not exist is \( x = 0, \) but this number is not from the domain of \( f. \) Therefore \( f \) does not have critical numbers.

**Step 2:** Since \( f \) does not have critical numbers, we evaluate it only at the endpoints \( x = 0.4 \) and \( x = 5. \) We have

\[
f(0.4) = -\frac{1}{(0.4)^2} = -\frac{1}{0.16} = -6.25
\]

\[
f(5) = -\frac{1}{5^2} = -\frac{1}{25} = -0.04
\]

**Step 3:** The largest value is -0.04 and the smallest value is -6.25. Therefore the absolute maximum of \( f \) on \([0.4, 5]\) is -0.04, occurring at \( x = 5 \) and the absolute minimum of \( f \) on \([0.4, 5]\) is -6.25, occurring at \( x = 0.4. \)
EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = 20 - 3x - \frac{12}{x} \) on \([2, 4]\) and determine where these values occur.

Solution:

**Step 1:** We have

\[
 f'(x) = \left(20 - 3x - \frac{12}{x}\right)' = (20 - 3x - 12x^{-1})' = 20' - (3x)' - (12x^{-1})' \\
= 20' - 3 \cdot x' - 12 \left(x^{-1}\right)'
\]

\[
= 0 - 3 \cdot 1 - 12(-1)x^{-1-1}
\]

\[
= -3 + 12x^{-2}
\]

Since

\[
-3 + 12x^{-2} = -3 + 12 \cdot \frac{1}{x^2} = -3 + \frac{12}{x^2} = \frac{-3x^2 + 12}{x^2} = \frac{-3(x^2 - 4)}{x^2} = \frac{-3(x^2 - 2^2)}{x^2} = \frac{-3(x - 2)(x + 2)}{x^2}
\]

we have

\[
 f'(x) = -\frac{3(x - 2)(x + 2)}{x^2}
\]

The numbers at which \( f' \) is zero are \( x = 2 \) and \( x = -2 \). The number at which \( f' \) does not exist is \( x = 0 \). But \( f \) is not defined at \( x = 0 \), therefore the critical numbers of \( f \) are \( x = -2 \) and \( x = 2 \) only.

**Step 2:** Since \(-2 \not\in [2, 4]\), we evaluate \( f \) only at the critical number \( x = 2 \) (which is also one of the endpoints) and at the second endpoint \( x = 4 \). We have

\[
 f(2) = 20 - 3 \cdot 2 - \frac{12}{2} = 20 - 6 - 6 = 8
\]

\[
 f(4) = 20 - 3 \cdot 4 - \frac{12}{4} = 20 - 12 - 3 = 5
\]

**Step 3:** The largest value is 8 and the smallest value is 5. Therefore the absolute maximum of \( f \) on \([2, 4]\) is 8, occurring at \( x = 2 \) and the absolute minimum of \( f \) on \([2, 4]\) is 5, occurring at \( x = 4 \).
EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = 2\sqrt{x} \) on \([-8, 1]\) and determine where these values occur.

Solution:

**Step 1:** We have

\[
 f'(x) = \left(2x^{1/3}\right)' = 2 \left(x^{1/3}\right)' = 2 \cdot \frac{1}{3} x^{1/3-1} = \frac{2}{3} x^{-2/3} = \frac{2}{3x^{2/3}}
\]

Note that there are no numbers at which \( f' \) is zero. The number at which \( f' \) does not exist is \( x = 0 \), so the critical number is \( x = 0 \).

**Step 2:** We evaluate \( f \) at the critical number \( x = 0 \) and at the endpoints \( x = -8 \) and \( x = 1 \). We have

\[
 f(0) = 2\sqrt{0} = 2 \cdot 0 = 0 \\
 f(-8) = 2\sqrt{-8} = 2 \cdot (-2) = -4 \\
 f(1) = 2\sqrt{1} = 2 \cdot 1 = 2
\]

**Step 3:** The largest value is 2 and the smallest value is \(-4\). Therefore the absolute maximum of \( f \) on \([-8, 1]\) is 2, occurring at \( x = 1 \) and the absolute minimum of \( f \) on \([-8, 1]\) is \(-4\), occurring at \( x = -8 \).

EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = -5x^{2/3} \) on \([-1, 1]\) and determine where these values occur.

Solution:

**Step 1:** We have

\[
 f'(x) = \left(-5x^{2/3}\right)' = -5 \left(x^{2/3}\right)' = -5 \cdot \frac{2}{3} x^{2/3-1} = -\frac{10}{3} x^{-1/3} = -\frac{10}{3x^{1/3}}
\]

Note that there are no numbers at which \( f' \) is zero. The number at which \( f' \) does not exist is \( x = 0 \), so the critical number is \( x = 0 \).

**Step 2:** We evaluate \( f \) at the critical number \( x = 0 \) and at the endpoints \( x = -1 \) and \( x = 1 \). We have

\[
 f(0) = -5(0)^{2/3} = -5 \cdot 0 = 0 \\
 f(-1) = -5(-1)^{2/3} = -5\left((-1)^2\right)^{1/3} = -5(1)^{1/3} = -5 \cdot 1 = -5 \\
 f(1) = -5(1)^{2/3} = -5 \cdot 1 = -5
\]

**Step 3:** The largest value is 0 and the smallest value is \(-5\). Therefore the absolute maximum of \( f \) on \([-1, 1]\) is 0, occurring at \( x = 0 \) and the absolute minimum of \( f \) on \([-1, 1]\) is \(-5\), occurring at \( x = \pm 1 \).

EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = x^{4/3} \) on \([-27, 27]\) and determine where these values occur.
EXAMPLE: Find the absolute maximum and minimum values of $f(x) = x^{4/3}$ on $[-27, 27]$ and determine where these values occur.

Solution:

**Step 1:** We have

$$f'(x) = \left(x^{4/3}\right)' = \frac{4}{3} x^{4/3 - 1} = \frac{4}{3} x^{1/3}$$

Note that there are no numbers at which $f'$ does not exist. The number at which $f'$ is zero is $x = 0$, so the critical number is $x = 0$.

**Step 2:** We evaluate $f$ at the critical number $x = 0$ and at the endpoints $x = -27$ and $x = 27$. We have

$$f(0) = 0^{4/3} = 0$$

$$f(-27) = (-27)^{4/3} = \left((-27)^{1/3}\right)^4 = (-3)^4 = 81$$

$$f(27) = 27^{4/3} = \left(27^{1/3}\right)^4 = 3^4 = 81$$

**Step 3:** The largest value is 81 and the smallest value is 0. Therefore the absolute maximum of $f$ on $[-27, 27]$ is 81, occurring at $x = \pm 27$ and the absolute minimum of $f$ on $[-27, 27]$ is 0, occurring at $x = 0$.

EXAMPLE: Find the absolute maximum and minimum values of $f(x) = \sqrt{4 - x^2}$ on $[-2, 1]$ and determine where these values occur.

Solution:

**Step 1:** We have

$$f'(x) = ((4 - x^2)^{1/2})' = \frac{1}{2} (4 - x^2)^{1/2 - 1} \cdot (4 - x^2)' = \frac{1}{2} (4 - x^2)^{-1/2} \cdot (4 - (x^2)')$$

$$= \frac{1}{2} (4 - x^2)^{-1/2} \cdot (0 - 2x) = \frac{1}{2} (4 - x^2)^{-1/2} \cdot (-2x) = -x(4 - x^2)^{-1/2}$$

$$= -\frac{x}{\sqrt{4 - x^2}}$$

Note that the number at which $f'$ is zero is $x = 0$ and the numbers at which $f'$ does not exist are $x = \pm 2$. Therefore the critical numbers are $x = 0, -2, 2$.

**Step 2:** Since $2 \notin [-2, 1]$, we evaluate $f$ only at the critical numbers $x = 0, -2$ (which is also one of the endpoints) and at the second endpoint $x = 1$. We have

$$f(0) = \sqrt{4 - 0^2} = \sqrt{4 - 0} = \sqrt{4} = 2$$

$$f(-2) = \sqrt{4 - (-2)^2} = \sqrt{4 - 4} = \sqrt{0} = 0$$

$$f(1) = \sqrt{4 - 1^2} = \sqrt{4 - 1} = \sqrt{3}$$

**Step 3:** The largest value is 2 and the smallest value is 0. Therefore the absolute maximum of $f$ on $[-2, 1]$ is 2, occurring at $x = 0$ and the absolute minimum of $f$ on $[-2, 1]$ is 0, occurring at $x = -2$. 

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EXAMPLE: Find the absolute maximum and minimum values of \( f(x) = \cos x \) on \( \left[ -\frac{\pi}{2}, \frac{5\pi}{2} \right] \) and determine where these values occur.

Solution:

**Step 1:** We have

\[ f'(x) = -\sin x \]

Note that there are no numbers at which \( f' \) does not exist. The numbers in \( \left[ -\frac{\pi}{2}, \frac{5\pi}{2} \right] \) at which \( f' \) is zero on are \( x = 0, \pi, 2\pi \), so the critical number is \( x = 0, \pi, 2\pi \).

**Step 2:** We evaluate \( f \) at the critical numbers \( x = 0, \pi, 2\pi \) and at the endpoints \( x = -\frac{\pi}{2}, \frac{5\pi}{2} \).

We have

\[
\begin{align*}
  f(0) &= \cos 0 = 1 \\
  f(\pi) &= \cos \pi = -1 \\
  f(2\pi) &= \cos(2\pi) = 1 \\
  f \left( -\frac{\pi}{2} \right) &= \cos \left( -\frac{\pi}{2} \right) = 0 \\
  f \left( \frac{5\pi}{2} \right) &= \cos \left( \frac{5\pi}{2} \right) = 0 
\end{align*}
\]

**Step 3:** The largest value is 1 and the smallest value is \(-1\). Therefore the absolute maximum of \( f \) on \( \left[ -\frac{\pi}{2}, \frac{5\pi}{2} \right] \) is 1, occurring at \( x = 0, 2\pi \) and the absolute minimum of \( f \) on \( \left[ -\frac{\pi}{2}, \frac{5\pi}{2} \right] \) is \(-1\), occurring at \( x = \pi \).