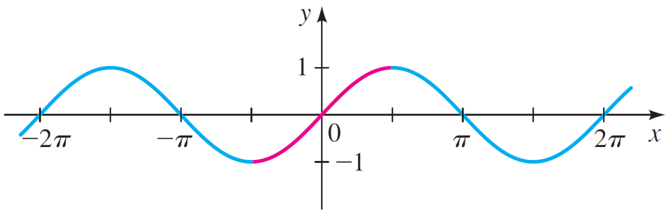
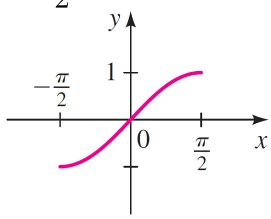
Inverse Trigonometric Functions

DEFINITION: The **inverse sine function,** denoted by $\sin^{-1} x$ (or $\arcsin x$), is defined to be the inverse of the restricted sine function

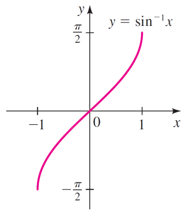
$$\sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$



$y = \sin x$

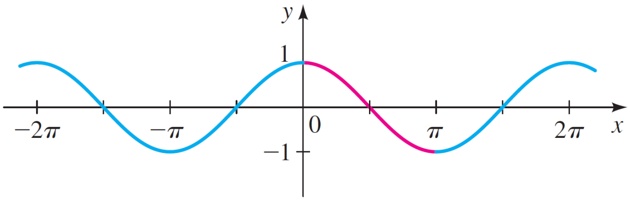


$y = \sin x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

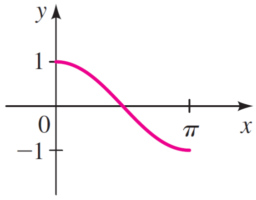


DEFINITION: The **inverse cosine function,** denoted by $\cos^{-1} x$ (or $\arccos x$), is defined to be the inverse of the restricted cosine function

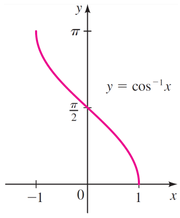
$$\cos x, \quad 0 \leq x \leq \pi$$



$y = \cos x$

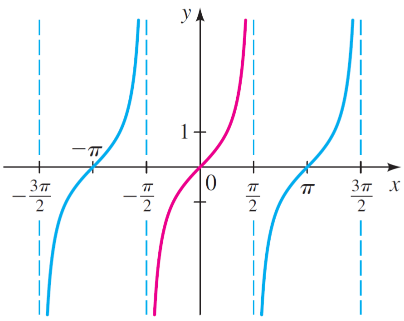


$y = \cos x,\ 0 \leq x \leq \pi$

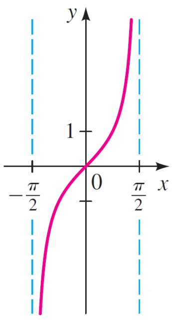


DEFINITION: The **inverse tangent function,** denoted by $\tan^{-1} x$ (or $\arctan x$), is defined to be the inverse of the restricted tangent function

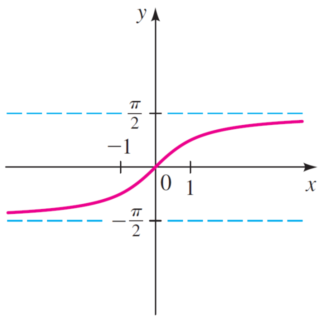
$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$



$y = \tan x$



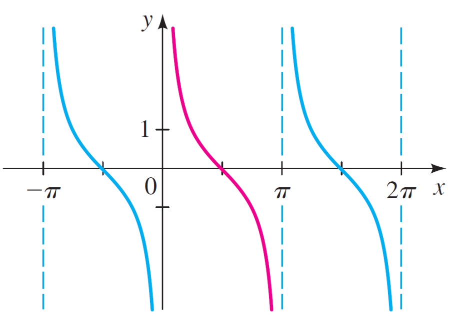
$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



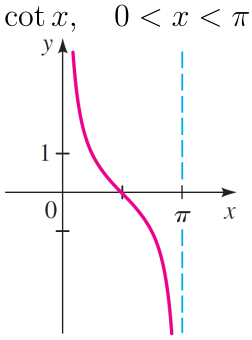
$y = \tan^{-1} x$

DEFINITION: The **inverse cotangent function,** denoted by $\cot^{-1} x$ (or $\operatorname{arccot} x$), is defined to be the inverse of the restricted cotangent function

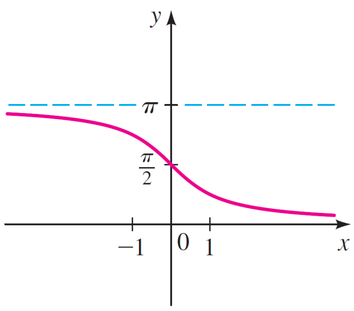
$$\cot x, \quad 0 < x < \pi$$



$y = \cot x$



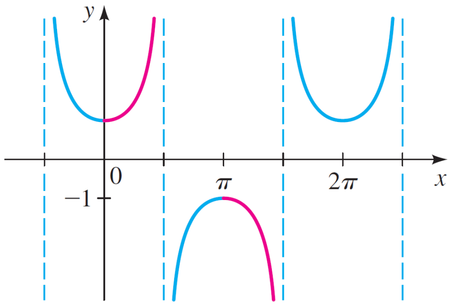
$y = \cot x,\ 0 < x < \pi$



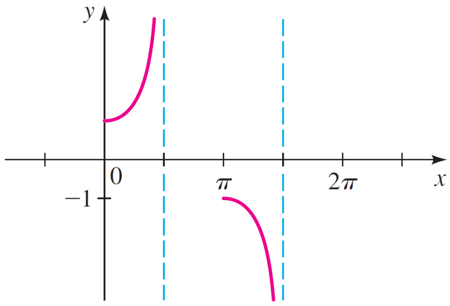
$y = \cot^{-1} x$

DEFINITION: The **inverse secant function,** denoted by $\sec^{-1} x$ (or $\operatorname{arcsec} x$), is defined to be the inverse of the restricted secant function

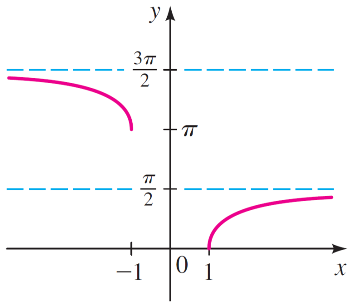
$$\sec x, \quad x \in [0, \pi/2) \cup [\pi, 3\pi/2) \quad \left[\text{or } x \in [0, \pi/2) \cup (\pi/2, \pi] \text{ in some other textbooks}\right]$$



$y = \sec x$



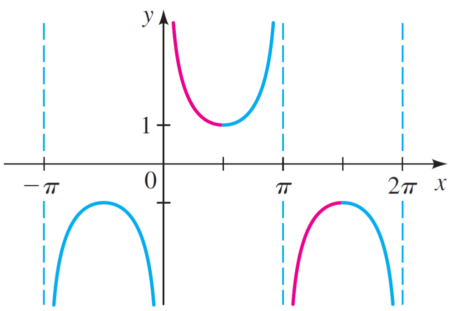
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$



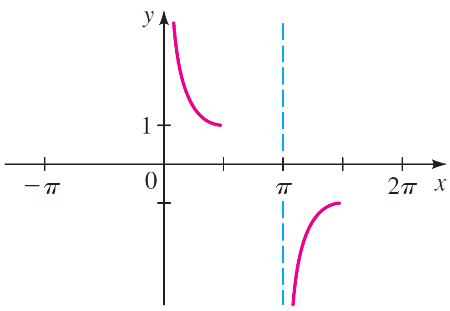
$y = \sec^{-1} x$

DEFINITION: The **inverse cosecant function,** denoted by $\csc^{-1} x$ (or $\operatorname{arccsc} x$), is defined to be the inverse of the restricted cosecant function

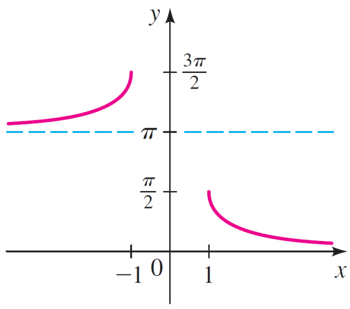
$$\csc x, \quad x \in (0, \pi/2] \cup (\pi, 3\pi/2] \quad \left[\text{or } x \in [-\pi/2, 0) \cup (0, \pi/2] \text{ in some other textbooks}\right]$$



$y = \csc x$



$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$



$y = \csc^{-1} x$

IMPORTANT: Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \cot^{-1} x, \quad \sec^{-1} x, \quad \csc^{-1} x$$

with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\cot x}, \quad \frac{1}{\sec x}, \quad \frac{1}{\csc x}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$	$\sec(-x) = \sec x$	$\cot(-x) = -\cot x$
$\sin(x \pm \pi) = -\sin x$	$\cos(x \pm \pi) = -\cos x$	$\tan(x \pm \pi) = \tan x$
$\sec(x \pm \pi) = -\sec x$	$\csc(x \pm \pi) = -\csc x$	$\cot(x \pm \pi) = \cot x$

EXAMPLES:

(a) $\sin^{-1} 1 = \frac{\pi}{2}$, since $\sin \frac{\pi}{2} = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\sin^{-1}(-1) = -\frac{\pi}{2}$, since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) $\sin^{-1} 0 = 0$, since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(d) $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$, since $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(e) $\sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$, since $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(f) $\sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$, since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLES:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi, \quad \cos^{-1} \frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1} \sqrt{3} = \frac{\pi}{3}, \quad \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1} x$	$(-\infty, +\infty)$	$(0, \pi)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup [\pi, 3\pi/2)$
$\csc^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$(0, \pi/2] \cup (\pi, 3\pi/2]$

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$	$\sec(-x) = \sec x$	$\cot(-x) = -\cot x$
$\sin(x \pm \pi) = -\sin x$	$\cos(x \pm \pi) = -\cos x$	$\tan(x \pm \pi) = \tan x$
$\sec(x \pm \pi) = -\sec x$	$\csc(x \pm \pi) = -\csc x$	$\cot(x \pm \pi) = \cot x$

EXAMPLES: Find $\sec^{-1} 1$, $\sec^{-1}(-1)$, and $\sec^{-1}(-2)$.

Solution: We have

$$\sec^{-1} 1 = 0, \qquad \sec^{-1}(-1) = \pi, \qquad \sec^{-1}(-2) = \frac{4\pi}{3}$$

since

$$\sec 0 = 1, \qquad \sec \pi = -1, \qquad \sec \frac{4\pi}{3} = -2$$

and

$$0, \pi, \frac{4\pi}{3} \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

Note that $\sec \dfrac{2\pi}{3}$ is also -2, but

$$\sec^{-1}(-2) \neq \frac{2\pi}{3}$$

since

$$\frac{2\pi}{3} \notin \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

EXAMPLES: Find

$$\tan^{-1} 0 \qquad \cot^{-1} 0 \qquad \cot^{-1} 1 \qquad \sec^{-1} \sqrt{2} \qquad \csc^{-1} 2 \qquad \csc^{-1} \frac{2}{\sqrt{3}}$$

EXAMPLES: We have

$$\tan^{-1} 0 = 0, \quad \cot^{-1} 0 = \frac{\pi}{2}, \quad \cot^{-1} 1 = \frac{\pi}{4}, \quad \sec^{-1} \sqrt{2} = \frac{\pi}{4}, \quad \csc^{-1} 2 = \frac{\pi}{6}, \quad \csc^{-1} \frac{2}{\sqrt{3}} = \frac{\pi}{3}$$

EXAMPLES: Evaluate $\sin\left(\arcsin\frac{\pi}{7}\right)$, $\arcsin\left(\sin\frac{\pi}{7}\right)$, and $\arcsin\left(\sin\frac{8\pi}{7}\right)$.

Solution: Since $\arcsin x$ is the inverse of the restricted sine function, we have

$$\sin(\arcsin x) = x \text{ if } x \in [-1,1] \qquad \text{and} \qquad \arcsin(\sin x) = x \text{ if } x \in [-\pi/2, \pi/2]$$

Therefore

$$\sin\left(\arcsin\frac{\pi}{7}\right) = \frac{\pi}{7} \qquad \text{and} \qquad \arcsin\left(\sin\frac{\pi}{7}\right) = \frac{\pi}{7}$$

but

$$\arcsin\left(\sin\frac{8\pi}{7}\right) = \arcsin\left(\sin\left(\frac{\pi}{7}+\pi\right)\right) = \arcsin\left(-\sin\frac{\pi}{7}\right) = -\arcsin\left(\sin\frac{\pi}{7}\right) = -\frac{\pi}{7}$$

EXAMPLES: Evaluate $\cot\left(\arcsin\frac{2}{5}\right)$ and $\sec\left(\arcsin\frac{2}{5}\right)$.

Solution 1: We have

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\pm\sqrt{1-\sin^2\theta}}{\sin\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\pm\sqrt{1-\sin^2\theta}}$$

Since $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$, it follows that $\cos(\arcsin x) \ge 0$. Therefore if $\theta = \arcsin\frac{2}{5}$, then

$$\cot\theta = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\sqrt{1-\sin^2\theta}}$$

hence

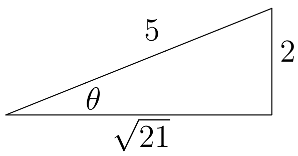
$$\cot\left(\arcsin\frac{2}{5}\right) = \frac{\sqrt{1-\sin^2\left(\arcsin\frac{2}{5}\right)}}{\sin\left(\arcsin\frac{2}{5}\right)} = \frac{\sqrt{1-\left(\frac{2}{5}\right)^2}}{\frac{2}{5}} = \frac{\sqrt{21}}{2}$$

and

$$\sec\left(\arcsin\frac{2}{5}\right) = \frac{1}{\sqrt{1-\sin^2\left(\arcsin\frac{2}{5}\right)}} = \frac{1}{\sqrt{1-\left(\frac{2}{5}\right)^2}} = \frac{5}{\sqrt{21}}$$

Solution 2: Put $\theta = \arcsin\frac{2}{5}$, so $\sin\theta = \frac{2}{5}$. Then

$$\cot\left(\arcsin\frac{2}{5}\right) = \cot\theta = \frac{\sqrt{21}}{2} \quad \text{and} \quad \sec\left(\arcsin\frac{2}{5}\right) = \sec\theta = \frac{5}{\sqrt{21}}$$



EXAMPLES: Evaluate, if possible, $\cot\left(\sin^{-1} 2\right)$ and $\sin\left(\tan^{-1} 2\right)$.

EXAMPLES: Evaluate, if possible, $\cot\left(\sin^{-1} 2\right)$ and $\sin\left(\tan^{-1} 2\right)$.

We first note that $\sin^{-1} 2$ does not exist, since $2 \notin [-1, 1]$, that is, 2 is not in the domain of $\sin^{-1} x$. Therefore $\cot\left(\sin^{-1} 2\right)$ does not exist.

We will evaluate $\sin\left(\tan^{-1} 2\right)$ in two different ways:
Solution 1: We have

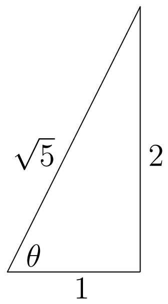
$$\sin\theta = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

Since $-\pi/2 < \tan^{-1} x < \pi/2$, it follows that $\cos\left(\tan^{-1} x\right) > 0$. Therefore if $\theta = \tan^{-1} 2$, then

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

hence

$$\sin\left(\tan^{-1} 2\right) = \frac{\tan\left(\tan^{-1} 2\right)}{\sqrt{1+\tan^2\left(\tan^{-1} 2\right)}} = \frac{2}{\sqrt{1+2^2}} = \frac{2}{\sqrt{5}}$$



Solution 2: Put $\theta = \tan^{-1} 2 = \tan^{-1}\dfrac{2}{1}$, so $\tan\theta = \dfrac{2}{1}$. Then

$$\sin\left(\tan^{-1} 2\right) = \sin\theta = \frac{2}{\sqrt{5}}$$

EXAMPLES: Evaluate $\sin\left(\cot^{-1}\left(-\dfrac{1}{2}\right)\right)$ and $\cos\left(\cot^{-1}\left(-\dfrac{1}{2}\right)\right)$.

EXAMPLES: Evaluate $\sin\left(\cot^{-1}\left(-\dfrac{1}{2}\right)\right)$ and $\cos\left(\cot^{-1}\left(-\dfrac{1}{2}\right)\right)$.

Solution 1: We have

$$\sin\theta=\pm\frac{1}{\sqrt{1+\cot^2\theta}} \quad \text{and} \quad \cos\theta=\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$$

Since $0<\cot^{-1}x<\pi$, it follows that $\sin(\cot^{-1}x)>0$. Therefore if $\theta=\cot^{-1}\left(-\dfrac{1}{2}\right)$, then

$$\sin\theta=\frac{1}{\sqrt{1+\cot^2\theta}} \quad \text{and} \quad \cos\theta=\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$$

hence

$$\sin\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)=\frac{1}{\sqrt{1+\cot^2\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)}}=\frac{1}{\sqrt{1+\left(-\frac{1}{2}\right)^2}}=\frac{2}{\sqrt{5}}$$

and

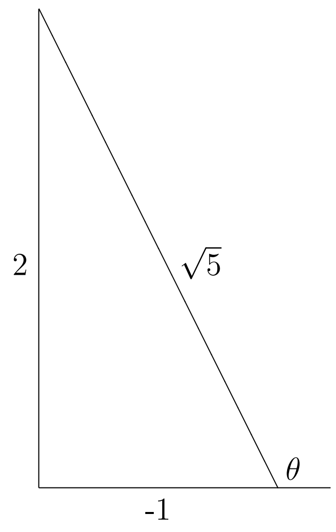
$$\cos\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)=\frac{\cot\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)}{\sqrt{1+\cot^2\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)}}=\frac{-\frac{1}{2}}{\sqrt{1+\left(-\frac{1}{2}\right)^2}}=-\frac{1}{\sqrt{5}}$$

Solution 2: Put $\theta=\cot^{-1}\left(-\dfrac{1}{2}\right)$, so $\cot\theta=-\dfrac{1}{2}=\dfrac{-1}{2}$. Then

$$\sin\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)=\sin\theta=\frac{2}{\sqrt{5}}$$

and

$$\cos\left(\cot^{-1}\left(-\frac{1}{2}\right)\right)=\cos\theta=-\frac{1}{\sqrt{5}}$$



THEOREM: We have

(a) $(\sin^{-1} u)' = \dfrac{1}{\sqrt{1-u^2}}u'$	(d) $(\cot^{-1} u)' = -\dfrac{1}{1+u^2}u'$
(b) $(\cos^{-1} u)' = -\dfrac{1}{\sqrt{1-u^2}}u'$	(e) $(\sec^{-1} u)' = \dfrac{1}{u\sqrt{u^2-1}}u'$
(c) $(\tan^{-1} u)' = \dfrac{1}{1+u^2}u'$	(f) $(\csc^{-1} u)' = -\dfrac{1}{u\sqrt{u^2-1}}u'$

Proof:

(a) Let $y = \sin^{-1} u$, then $\sin y = u$. Therefore

$$(\sin y)' = u' \quad \Longrightarrow \quad \cos y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\cos y}$$

Since $-\dfrac{\pi}{2} \leq \underbrace{\sin^{-1} u}_{y} \leq \dfrac{\pi}{2}$, it follows that $\cos y \geq 0$. Hence

$$\cos y = \sqrt{1 - \sin^2 y} = [\sin y = u] = \sqrt{1-u^2} \quad \Longrightarrow \quad y' = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}$$

(b) Let $y = \cos^{-1} u$, then $\cos y = u$. Therefore

$$(\cos y)' = u' \quad \Longrightarrow \quad -\sin y \cdot y' = u' \quad \Longrightarrow \quad y' = -\frac{u'}{\sin y}$$

Since $0 \leq \underbrace{\cos^{-1} u}_{y} \leq \pi$, it follows that $\sin y \geq 0$. Hence

$$\sin y = \sqrt{1 - \cos^2 y} = [\cos y = u] = \sqrt{1-u^2} \quad \Longrightarrow \quad y' = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}$$

(c) Let $y = \tan^{-1} u$, then $\tan y = u$. Therefore

$$(\tan y)' = u' \quad \Longrightarrow \quad \sec^2 y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\sec^2 y}$$

Note, that $\sec^2 y = 1 + \tan^2 y = [\tan y = u] = 1 + u^2$. Hence

$$y' = \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}$$

(a) $(\sin^{-1} u)' = \dfrac{1}{\sqrt{1-u^2}}u'$	(d) $(\cot^{-1} u)' = -\dfrac{1}{1+u^2}u'$
(b) $(\cos^{-1} u)' = -\dfrac{1}{\sqrt{1-u^2}}u'$	(e) $(\sec^{-1} u)' = \dfrac{1}{u\sqrt{u^2-1}}u'$
(c) $(\tan^{-1} u)' = \dfrac{1}{1+u^2}u'$	(f) $(\csc^{-1} u)' = -\dfrac{1}{u\sqrt{u^2-1}}u'$

(d) Let $y = \cot^{-1} u$, then $\cot y = u$. Therefore

$$(\cot y)' = u' \quad \Longrightarrow \quad -\csc^2 y \cdot y' = u' \quad \Longrightarrow \quad y' = -\frac{u'}{\csc^2 y}$$

Note, that $\csc^2 y = 1 + \cot^2 y = [\cot y = u] = 1 + u^2$. Hence

$$y' = -\frac{u'}{\csc^2 y} = -\frac{u'}{1+u^2}$$

(e) Let $y = \sec^{-1} u$, then $\sec y = u$. Therefore

$$(\sec y)' = u' \quad \Longrightarrow \quad \sec y \tan y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\sec y \tan y}$$

Since $\underbrace{\sec^{-1} u}_{y} \in [0, \pi/2) \cup [\pi, 3\pi/2)$, it follows that $\tan y \geq 0$. Hence

$$\sec y \tan y = \sec y \sqrt{\sec^2 y - 1} = [\sec y = u] = u\sqrt{u^2-1} \quad \Longrightarrow \quad y' = \frac{u'}{\sec y \tan y} = \frac{u'}{u\sqrt{u^2-1}}$$

(f) Let $y = \csc^{-1} u$, then $\csc y = u$. Therefore

$$(\csc y)' = u' \quad \Longrightarrow \quad -\csc y \cot y \cdot y' = u' \quad \Longrightarrow \quad y' = -\frac{u'}{\csc y \cot y}$$

Since $\underbrace{\csc^{-1} u}_{y} \in (0, \pi/2] \cup (\pi, 3\pi/2]$, it follows that $\cot y \geq 0$. Hence

$$\csc y \cot y = \csc y \sqrt{\csc^2 y - 1} = [\csc y = u] = u\sqrt{u^2-1} \quad \Longrightarrow \quad y' = -\frac{u'}{\csc y \cot y} = -\frac{u'}{u\sqrt{u^2-1}}$$

EXAMPLES:

(a) Let $f(x) = x \tan^{-1}(1-2x)$. Find $f'(x)$.

(b) Let $f(x) = 2^{\sin^{-1}(4x)}$. Find $f'(x)$.

(c) Let $f(x) = \sqrt{\sec^{-1}(1-3x)}$. Find $f'(x)$.

(a) $(\sin^{-1} u)' = \dfrac{1}{\sqrt{1-u^2}}u'$

(b) $(\cos^{-1} u)' = -\dfrac{1}{\sqrt{1-u^2}}u'$

(c) $(\tan^{-1} u)' = \dfrac{1}{1+u^2}u'$

(d) $(\cot^{-1} u)' = -\dfrac{1}{1+u^2}u'$

(e) $(\sec^{-1} u)' = \dfrac{1}{u\sqrt{u^2-1}}u'$

(f) $(\csc^{-1} u)' = -\dfrac{1}{u\sqrt{u^2-1}}u'$

EXAMPLES:

(a) Let $f(x) = x\tan^{-1}(1-2x)$. Then

$$\begin{aligned} f'(x) = [x\tan^{-1}(1-2x)]' &= x'\tan^{-1}(1-2x) + x[\tan^{-1}(1-2x)]' \\ &= 1\cdot\tan^{-1}(1-2x) + x\frac{1}{1+(1-2x)^2}\cdot(1-2x)' \\ &= \tan^{-1}(1-2x) + x\frac{1}{1+(1-2x)^2}\cdot(-2) \\ &= \tan^{-1}(1-2x) - \frac{2x}{1+(1-2x)^2} \\ &= \tan^{-1}(1-2x) - \frac{x}{1-2x+2x^2} \end{aligned}$$

(b) Let $f(x) = 2^{\sin^{-1}(4x)}$. Then

$$\begin{aligned} f'(x) = \left[2^{\sin^{-1}(4x)}\right]' &= 2^{\sin^{-1}(4x)}\ln 2\cdot\left[\sin^{-1}(4x)\right]' \\ &= 2^{\sin^{-1}(4x)}\ln 2\frac{1}{\sqrt{1-(4x)^2}}\cdot(4x)' \\ &= 2^{\sin^{-1}(4x)}\ln 2\frac{1}{\sqrt{1-(4x)^2}}\cdot 4 \\ &= \frac{2^{\sin^{-1}(4x)+2}\ln 2}{\sqrt{1-(4x)^2}} \end{aligned}$$

(c) Let $f(x) = \sqrt{\sec^{-1}(1-3x)}$. Then

$$\begin{aligned} f'(x) = [(\sec^{-1}(1-3x))^{1/2}]' &= \frac{1}{2}(\sec^{-1}(1-3x))^{-1/2}\cdot[\sec^{-1}(1-3x)]' \\ &= \frac{1}{2}(\sec^{-1}(1-3x))^{-1/2}\frac{1}{(1-3x)\sqrt{(1-3x)^2-1}}\cdot(1-3x)' \\ &= \frac{1}{2}(\sec^{-1}(1-3x))^{-1/2}\frac{1}{(1-3x)\sqrt{(1-3x)^2-1}}\cdot(-3) \\ &= -\frac{3}{2(1-3x)\sqrt{3x(3x-2)\sec^{-1}(1-3x)}} \end{aligned}$$