Exponential Growth and Decay

In many natural phenomena (such as population growth, radioactive decay, etc.), quantities grow or decay at a rate proportional to their size. In other words, they satisfy the following **differential equation**

$$\boxed{\frac{dy}{dt} = ky, \quad \text{where } k \text{ is a constant}} \tag{1}$$

If $k > 0$, we call it the **law of natural growth**. If $k < 0$, we call it the **law of natural decay**.

THEOREM: The only solution of the differential equation (1) are the exponential functions

$$\boxed{y(t) = y(0)e^{kt}} \tag{2}$$

REMARK: It is easy to check that (2) satisfies (1). In fact,

$$\frac{dy}{dt} = (y(0)e^{kt})' = y(0)(e^{kt})' = y(0)e^{kt} \cdot (kt)' = y(0)e^{kt} \cdot k = k\underbrace{y(0)e^{kt}}_{y(t)} = ky(t)$$

Population Growth

EXAMPLE: Use the fact that the world population was 2560 million people in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the **relative growth rate** k? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Solution: Since the growth rate is proportional to the population size, we have

$$\frac{dP}{dt} = kP$$

and therefore by the Theorem above we get

$$P(t) = P(0)e^{kt} \tag{3}$$

To find k we note that $P(0) = 2560$ and $P(10) = 3040$. Using this in (3) with $t = 10$, we get

$$P(10) = P(0)e^{kt} \quad \Longrightarrow \quad 3040 = 2560e^{k\cdot 10} \quad \Longrightarrow \quad \frac{3040}{2560} = e^{k\cdot 10}$$

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If we logarithm both sides, we get

$$\ln \frac{3040}{2560} = \ln(e^{k\cdot 10}) = 10k \ln e = 10k \quad \Longrightarrow \quad k = \frac{1}{10} \ln \frac{3040}{2560} \approx 0.017185$$

Substituting this into (3) and keeping in mind that $P(0) = 2560$, we obtain

$$P(t) = 2560e^{0.017185t} \tag{4}$$

Equation (4) helps us to estimate the world population in 1993 and to predict the population in the year 2020. In fact, the world population in 1993 was

$$P(43) = 2560e^{0.017185\cdot 43} \approx 5360 \text{ million (actual is 5522 million)}$$

Similarly, the world population in 2020 will be

$$P(70) = 2560e^{0.017185\cdot 70} \approx 8524 \text{ million}$$

EXAMPLE: At the start of an experiment, there are 100 bacteria. If the bacteria follow an exponential growth pattern with rate $k = 0.02$, what will be the population after 5 hours? How long will it take for the population to double?

EXAMPLE: At the start of an experiment, there are 100 bacteria. If the bacteria follow an exponential growth pattern with rate $k = 0.02$, what will be the population after 5 hours? How long will it take for the population to double?

Solution: By the Theorem above we get

$$P(t) = P(0)e^{kt}$$

where $P(0) = 100$ and $k = 0.02$. Therefore

$$P(5) = 100e^{0.02\cdot 5} \approx 110.517 \approx 110 \text{ bacteria}$$

The second question suggests the following equation

$$2 \cdot 100 = 100e^{0.02t} \quad \Longrightarrow \quad 2 = e^{0.02t}$$

If we logarithm both sides, we get

$$\ln 2 = \ln e^{0.02t} = 0.02t \ln e = 0.02t \quad \Longrightarrow \quad t = \frac{\ln 2}{0.02} \approx 34.6574 \text{ hours}$$

EXAMPLE: Suppose that the population of a colony of bacteria increases exponentially. At the start of an experiment, there are 6,000 bacteria, and one hour later, the population has increased to 6,400. How long will it take for the population to reach 10,000? Round your answer to the nearest hour.

Solution: By the Theorem above we get

$$P(t) = P(0)e^{kt}$$

where $P(0) = 6000,\ P(1) = 6400$ and $t = 1$. Therefore

$$6400 = 6000e^{k\cdot 1} \quad \Longrightarrow \quad \frac{6400}{6000} = e^k \quad \Longrightarrow \quad k = \ln\left(\frac{6400}{6000}\right) \approx 0.06454$$

so

$$P(t) = 6000e^{0.06454t}$$

The second question suggests the following equation

$$10000 = 6000e^{0.06454t} \quad \Longrightarrow \quad \frac{10000}{6000} = e^{0.06454t}$$

If we logarithm both sides, we get

$$\ln\left(\frac{10000}{6000}\right) = \ln e^{0.06454t} = 0.06454t \ln e = 0.06454t$$

therefore

$$t = \frac{1}{0.06454} \ln\left(\frac{10000}{6000}\right) \approx 7.915 \approx 8 \text{ hours}$$

Radioactive Decay

EXAMPLE: The half-life of radium-226 ($^{226}_{88}$Ra) is 1590 years.

(a) A sample of radium-226 has a mass of 100 mg. Find the formula for the mass of $^{226}_{88}$Ra that remains after t years.

(b) Find the mass after 1000 years correct to the nearest milligram.

(c) When will the mass be reduced to 30 mg?

Solution:
(a) Let $m(t)$ be the mass of radium-226 (in milligrams) that remains after t years. By the Theorem above we have

$$m(t) = m(0)e^{kt}$$

To find k we use this formula with $t = 1590$, $m(0) = 100$ and $m(1590) = \frac{1}{2}m(0) = \frac{1}{2} \cdot 100 = 50$. We have

$$100e^{1590k} = 50 \quad \Longrightarrow \quad e^{1590k} = \frac{50}{100} = \frac{1}{2}$$

If we logarithm both sides, we get

$$\ln\left(\frac{1}{2}\right) = \ln e^{1590k} = 1590k \ln e = 1590k$$

hence

$$k = \frac{1}{1590}\ln\left(\frac{1}{2}\right) = -\frac{\ln 2}{1590}$$

Therefore

$$m(t) = 100e^{-(\ln 2)t/1590}$$

(b)

(c)

(b) The mass after 1000 years is

$$m(1000) = 100e^{-(\ln 2)1000/1590} \approx 65 \text{ mg}$$

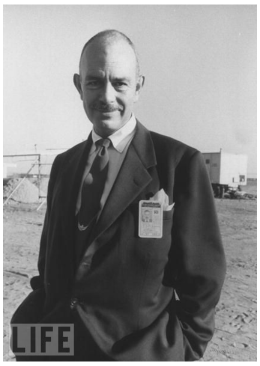
(c) To answer the last question, we should find the value of t such that $m(t) = 30$, that is,

$$100e^{-(\ln 2)t/1590} = 30 \quad \Longrightarrow \quad e^{-(\ln 2)t/1590} = 0.3$$

If we logarithm both sides, we get

$$\ln 0.3 = \ln e^{-(\ln 2)t/1590} = -\frac{\ln 2}{1590} t \ln e = -\frac{\ln 2}{1590} t \quad \Longrightarrow \quad t = -\frac{1590 \ln 0.3}{\ln 2} \approx 2762 \text{ years}$$

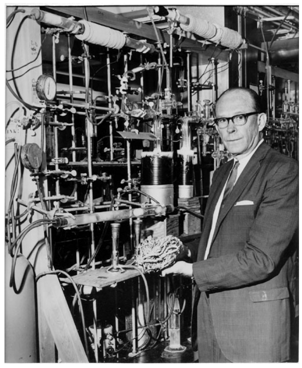
Wikipedia: Carbon-14 or radiocarbon, is a radioactive isotope of carbon discovered on February 27, 1940, by Martin Kamen and Samuel Ruben at the University of California Radiation Laboratory in Berkeley, though its existence had been suggested already in 1934 by Franz Kurie. Radiocarbon dating is a radiometric dating method that uses Carbon-14 to determine the age of carbonaceous materials up to about 60,000 years old. The technique was developed by Willard Libby and his colleagues in 1949 during his tenure as a professor at the University of Chicago. Libby estimated that the radioactivity of exchangeable carbon-14 would be about 14 disintegrations per minute (dpm) per gram. In 1960, he was awarded the Nobel Prize in chemistry for this work. One of the frequent uses of the technique is to date organic remains from archaeological sites.



Franz Kurie

Martin Kamen

Samuel Ruben



Willard Libby

EXAMPLE: You find a skull in a nearby Native American ancient burial site and with the help of a spectrometer, discover that the skull contains 9% of the C-14 found in a modern skull. Assuming that the half life of C-14 (radiocarbon) is 5730 years, how old is the skull?

Solution: Let us denote the initial amount of C-14 by C. After 5730 years we have half of this amount, which is $\frac{1}{2}C$. Using this in (2), we get

$$\frac{1}{2}C = Ce^{k \cdot 5730} \quad \Longrightarrow \quad \frac{1}{2} = e^{k \cdot 5730} \ldots$$

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$$\frac{1}{2}C = Ce^{k\cdot 5730} \quad \Longrightarrow \quad \frac{1}{2} = e^{k\cdot 5730}$$

If we logarithm both sides, we get

$$\ln\left(\frac{1}{2}\right) = \ln e^{k\cdot 5730} = 5730k \ln e = 5730k \quad \Longrightarrow \quad k = \frac{\ln 0.5}{5730} \approx -0.000121$$

Now we use the fact that there is 9% remaining today to give

$$0.09C = Ce^{-0.000121t} \quad \Longrightarrow \quad 0.09 = e^{-0.000121t}$$

If we logarithm both sides, we get

$$\ln 0.09 = \ln e^{-0.000121t} = -0.000121t \ln e = -0.000121t \quad \Longrightarrow \quad t = \frac{\ln 0.09}{-0.000121} \approx 19,905$$

So, the skull is about 20,000 years old.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large. (This law also applies to warming.) If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant. This equation is not quite the same as Equation 1, so we make the change of variable $y(t) = T(t) - T_s$. Because T_s is constant, we have $y'(t) = T'(t)$ and so the equation becomes

$$\frac{dy}{dt} = ky$$

We can then use (2) to find an expression for y, from which we can find T.

EXAMPLE: A bottle of soda pop at room temperature (72°F) is placed in a refrigerator where the temperature is 44°F. After half an hour the soda pop has cooled to 61°F.

(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

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(a) What is the temperature of the soda pop after another half hour?

(b) How long does it take for the soda pop to cool to 50°F?

Solution:

(a) Let $T(t)$ be the temperature of the soda after t minutes. The surrounding temperature is $T_s = 44$°F, so Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - 44)$$

If we let $y = T - 44$, then $y(0) = T(0) - 44 = 72 - 44 = 28$, so y satisfies

$$\frac{dy}{dt} = ky \qquad y(0) = 28$$

and by (2) we have

$$y(t) = y(0)e^{kt} = 28e^{kt}$$

We are given that $T(30) = 61$, so $y(30) = T(30) - 44 = 61 - 44 = 17$ and

$$28e^{30k} = 17 \quad \Longrightarrow \quad e^{30k} = \frac{17}{28}$$

If we logarithm both sides, we get

$$\ln\left(\frac{17}{28}\right) = \ln e^{30k} = 30k \ln e = 30k \quad \Longrightarrow \quad k = \frac{1}{30}\ln\left(\frac{17}{28}\right) \approx -0.01663$$

Thus

$$y(t) = 28e^{-001663t} \quad \Longrightarrow \quad T(t) = 44 + 28e^{-001663t} \quad \Longrightarrow \quad T(60) = 44 + 28e^{-001663\cdot 60} \approx 54.3$$

So after another half hour the pop has cooled to about 54°F.

(b) We have $T(t) = 50$ when

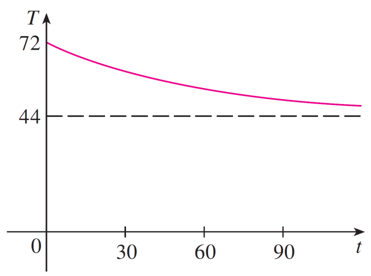
$$44 + 28e^{-001663t} = 50 \quad \Longrightarrow \quad e^{-001663t} = \frac{6}{28} \quad \Longrightarrow \quad t = \frac{1}{-0.01663}\ln\left(\frac{6}{28}\right) \approx 92.6$$

The pop cools to 50°F after about 1 hour 33 minutes.

REMARK: Notice that in the above Example, we have

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty}(44 + 28e^{-001663t}) = 44 + 28 \cdot 0 = 44$$

which is to be expected. Here is the graph of the temperature function:



Continuous Compounded Interest

If \$100 is invested at 2% interest, **compounded annually**, then after 1 year the investment is worth

$$\$100(1+0.02)=\$102$$

If \$100 is invested at 2% interest, **compounded semiannually**, then after 1 year the investment is worth

$$\$100\left(1+\frac{0.02}{2}\right)=\$101 \text{ (after first 6 months)}$$

$$\$101\left(1+\frac{0.02}{2}\right)=\$102.01 \text{ (after 1 year)}$$

The same result can be obtained in a more elegant way:

$$\$100\left(1+\frac{0.02}{2}\right)=\$101 \text{ (after first 6 months)}$$

$$\$100\left(1+\frac{0.02}{2}\right)\left(1+\frac{0.02}{2}\right)=\$100\left(1+\frac{0.02}{2}\right)^2=\$102.01 \text{ (after 1 year)}$$

If \$100 is invested at 2% interest, **compounded quarterly**, then after 1 year the investment is worth

$$\$100\left(1+\frac{0.02}{4}\right)=\$100.5 \text{ (after first 3 months)}$$

$$\$100.5\left(1+\frac{0.02}{4}\right)=\$101.0025 \text{ (after first 6 months)}$$

$$\$101.0025\left(1+\frac{0.02}{4}\right)=\$101.5075125 \text{ (after first 9 months)}$$

$$\$101.5075125\left(1+\frac{0.02}{4}\right)=\$102.0150500625 \text{ (after 1 year)}$$

As before, the same result can be obtained in a more elegant way:

$$\$100\left(1+\frac{0.02}{4}\right)=\$100.5 \text{ (after first 3 months)}$$

$$\$100\left(1+\frac{0.02}{4}\right)\left(1+\frac{0.02}{4}\right)=\$100\left(1+\frac{0.02}{4}\right)^2=\$101.0025 \text{ (after first 6 months)}$$

$$\$100\left(1+\frac{0.02}{4}\right)^2\left(1+\frac{0.02}{4}\right)=\$100\left(1+\frac{0.02}{4}\right)^3=\$101.5075125 \text{ (after first 9 months)}$$

$$\$100\left(1+\frac{0.02}{4}\right)^3\left(1+\frac{0.02}{4}\right)=\$100\left(1+\frac{0.02}{4}\right)^4=\$102.0150500625 \text{ (after 1 year)}$$

In general, if we invest A_0 dollars at interest r, compounded n times a year, then after 1 year the investment is worth

$$A_0\left(1+\frac{r}{n}\right)^n \text{ dollars}$$

Moreover, after t years the investment is worth

$$\boxed{A(t)=A_0\left(1+\frac{r}{n}\right)^{nt}} \tag{5}$$

QUESTION: What happens if $n\to\infty$?

Answer: We have

$$\begin{aligned} A(t) &= \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = \lim_{n\to\infty} A_0\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} = A_0\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt} \\ &= A_0\left[\lim_{n\to\infty}\left(1+\frac{1}{n/r}\right)^{n/r}\right]^{rt} = A_0\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{rt} = A_0e^{rt} \end{aligned}$$

so

$$\boxed{A(t)=A_0e^{rt}} \tag{6}$$

EXAMPLE: If \$100 is invested at 2% interest, **compounded continuously**, then after 1 year the investment is worth

$$A(1)=\$100e^{0.02\cdot 1}\approx \$102.02$$

EXAMPLE: If \$200,000 is borrowed at 5.5% interest, find the amounts due at the end of 30 years if the interest compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) continuously.

Solution:

(i) By (5) we have

$$A(30)=A_0\left(1+\frac{r}{n}\right)^{n\cdot 30}=\$200,000\left(1+\frac{0.055}{1}\right)^{1\cdot 30}\approx \$996,790.26$$

(ii) By (5) we have

$$A(30)=A_0\left(1+\frac{r}{n}\right)^{n\cdot 30}=\$200,000\left(1+\frac{0.055}{4}\right)^{4\cdot 30}\approx \$1,029,755.36$$

which gives $\approx \$32,965.10$ difference between (ii) and (i).

(iii) By (5) we have

$$A(30)=A_0\left(1+\frac{r}{n}\right)^{n\cdot 30}=\$200,000\left(1+\frac{0.055}{12}\right)^{12\cdot 30}\approx \$1,037,477.57$$

which gives $\approx \$7,722.21$ difference between (iii) and (ii).

(iv) By (6) we have

$$A(30)=A_0e^{r\cdot 30}=\$200,000e^{0.055\cdot 30}\approx \$1,041,395.97$$

which gives $\approx \$3,918.38$ difference between (iv) and (iii).

EXAMPLE: If $200,000 is borrowed at 5.6% interest, find the amounts due at the end of 30 years if the interest compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) continuously.

Solution:

(i) By (5) we have

$$A(30) = A_0\left(1+\frac{r}{n}\right)^{n\cdot 30} = \$200,000\left(1+\frac{0.056}{1}\right)^{1\cdot 30} \approx \$1,025,528.05$$

which gives $\approx \$28,737.79$ difference between 5.6% and 5.5%.

(ii) By (5) we have

$$A(30) = A_0\left(1+\frac{r}{n}\right)^{n\cdot 30} = \$200,000\left(1+\frac{0.056}{4}\right)^{4\cdot 30} \approx \$1,060,680.53$$

which gives $\approx \$35,152.48$ difference between (ii) and (i).

(iii) By (5) we have

$$A(30) = A_0\left(1+\frac{r}{n}\right)^{n\cdot 30} = \$200,000\left(1+\frac{0.056}{12}\right)^{12\cdot 30} \approx \$1,068,925.95$$

which gives $\approx \$8,245.43$ difference between (iii) and (ii).

(iv) By (6) we have

$$A(30) = A_0 e^{r\cdot 30} = \$200,000 e^{0.056\cdot 30} \approx \$1,073,111.19$$

which gives $\approx \$4,185.24$ difference between (iv) and (iii).