Derivatives of Logarithmic and Exponential Functions

THEOREM: The function $f(x) = \log_a x$ is differentiable and

$$\boxed{f'(x) = \frac{1}{x \ln a}}$$

Proof: We have:

$$\frac{d}{dx}(\log_a x) = \lim_{h\to 0} \frac{\log_a(x+h) - \log_a x}{h} = \lim_{h\to 0}\left[\frac{1}{h}\log_a\left(\frac{x+h}{x}\right)\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\log_a\left(1+\frac{h}{x}\right)\right] = \lim_{h\to 0}\left[\frac{1}{x}\cdot\frac{x}{h}\log_a\left(1+\frac{h}{x}\right)\right]$$

$$= \frac{1}{x}\lim_{h\to 0}\left[\log_a\left(1+\frac{h}{x}\right)^{x/h}\right] = \frac{1}{x}\lim_{h\to 0}\left[\log_a\left(1+\frac{h}{x}\right)^{1/(h/x)}\right]$$

$$= \frac{1}{x}\log_a\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/(h/x)}\right] = \left[\lim_{u\to 0}(1+u)^{1/u} = e\right] = \frac{1}{x}\log_a e = \frac{1}{x}\frac{\ln e}{\ln a} = \frac{1}{x\ln a}$$

COROLLARY: We have

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

REMARK: In general,

$$\boxed{(\log_a u)' = \frac{1}{u\ln a}\cdot u'} \quad \text{and} \quad \boxed{(\ln u)' = \frac{1}{u}\cdot u'}$$

EXAMPLES:

(a) If $f(x) = \log_5(x^2+1)$, then $f'(x) = [\log_5(x^2+1)]' = \dfrac{1}{(x^2+1)\ln 5}\cdot(x^2+1)' = \dfrac{2x}{(x^2+1)\ln 5}$.

(b) If $f(x) = \ln(\ln x)$, then $f'(x) = [\ln(\ln x)]' = \dfrac{1}{\ln x}\cdot(\ln x)' = \dfrac{1}{x\ln x}$.

(c) Find $f'(x)$ if $f(x) = \log\left(\dfrac{x}{1+x^2}\right)$.

(c) If $f(x)=\log\left(\dfrac{x}{1+x^2}\right)$, then

$$f'(x)=\left[\log\left(\frac{x}{1+x^2}\right)\right]'=\frac{1}{\frac{x}{1+x^2}\cdot\ln 10}\cdot\left(\frac{x}{1+x^2}\right)'$$

$$=\left\{\frac{(1+x^2)\cdot 1}{(1+x^2)\cdot\frac{x}{1+x^2}\cdot\ln 10}\cdot\left(\frac{x}{1+x^2}\right)'\right\}$$

$$=\frac{1+x^2}{x\ln 10}\cdot\left(\frac{x}{1+x^2}\right)'$$

$$=\frac{1+x^2}{x\ln 10}\cdot\frac{x'(1+x^2)-x(1+x^2)'}{(1+x^2)^2}$$

$$=\frac{1+x^2}{x\ln 10}\cdot\frac{1\cdot(1+x^2)-x\cdot 2x}{(1+x^2)^2}$$

$$=\frac{1+x^2}{x\ln 10}\cdot\frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$=\frac{1+x^2}{x\ln 10}\cdot\frac{1-x^2}{(1+x^2)^2}$$

$$=\frac{1-x^2}{x(1+x^2)\ln 10}$$

or

$$f'(x)=\left[\log\left(\frac{x}{1+x^2}\right)\right]'=\left[\log x-\log(1+x^2)\right]'$$

$$=\frac{1}{x\ln 10}-\frac{1}{(1+x^2)\ln 10}\cdot(1+x^2)'$$

$$=\frac{1}{x\ln 10}-\frac{2x}{(1+x^2)\ln 10}$$

$$=\frac{1+x^2}{x(1+x^2)\ln 10}-\frac{2x^2}{x(1+x^2)\ln 10}$$

$$=\frac{1+x^2-2x^2}{x(1+x^2)\ln 10}$$

$$=\frac{1-x^2}{x(1+x^2)\ln 10}$$

THEOREM: We have

$$\boxed{\frac{d}{dx}(a^x) = a^x \ln a}$$

In particular, if $a = e$, then

$$\boxed{(e^x)' = e^x}$$

Proof 1: Let $y = a^x$. We should prove that $y' = a^x \ln a$. We have

$$y = a^x \quad \Longrightarrow \quad \ln y = \ln a^x = x \ln a$$

therefore

$$(\ln y)' = (x \ln a)' = \ln a \cdot (x)' = \ln a \quad \Longrightarrow \quad \frac{1}{y} \cdot y' = \ln a$$

so

$$y' = y \ln a = a^x \ln a$$

Proof 2: Let $y = a^x$, then $\log_a y = x$. We have

$$(\log_a y)' = x' \quad \Longrightarrow \quad \frac{1}{y \ln a} \cdot y' = 1$$

so

$$y' = y \ln a = a^x \ln a$$

Proof 3: Let $f(x) = \log_a x$. Note that $f^{-1}(x) = a^x$. Since

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \text{and} \quad f'(x) = (\log_a x)' = \frac{1}{x \ln a}$$

we have

$$(a^x)' = \frac{1}{f'(f^{-1}(a))} = \frac{1}{\frac{1}{a^x \ln a}} = a^x \ln a$$

REMARK: In general,

$$\boxed{(a^u)' = a^u \ln a \cdot u'} \quad \text{and} \quad \boxed{(e^u)' = e^u \cdot u'}$$

EXAMPLES:

1. Find $(2^{\sin 3x})'$.

2. Find $(xe^{\sqrt{1-x}})'$.

EXAMPLES:

1. $(2^{\sin 3x})' = 2^{\sin 3x}\ln 2\cdot(\sin 3x)' = 2^{\sin 3x}\ln 2\cdot\cos 3x\cdot(3x)' = 2^{\sin 3x}\ln 2\cdot\cos 3x\cdot 3$

2. $(xe^{\sqrt{1-x}})' = x'e^{\sqrt{1-x}} + x(e^{\sqrt{1-x}})' = e^{\sqrt{1-x}} + xe^{\sqrt{1-x}}\cdot(\sqrt{1-x})'$

$$= e^{\sqrt{1-x}} + xe^{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{1-x}}(1-x)' = e^{\sqrt{1-x}} - \frac{xe^{\sqrt{1-x}}}{2\sqrt{1-x}}$$

Logarithmic Differentiation

Note that we can't apply the rules $(u^n)' = nu^{n-1}\cdot u'$ or $(a^u)' = a^u\ln a\cdot u'$ to functions like $f(x) = x^x$.

EXAMPLES:

1. Let $f(x) = x^x$. Find $f'(x)$.

Solution: We logarithm and then differentiate both sides of $f(x) = x^x$. We have

$$f(x) = x^x \quad\Longrightarrow\quad \ln f(x) = \ln x^x = x\ln x \quad\Longrightarrow\quad [\ln f(x)]' = [x\ln x]'$$

therefore

$$\frac{1}{f(x)}\cdot f'(x) = x'\ln x + x(\ln x)' = 1\cdot\ln x + x\cdot\frac{1}{x} = \ln x + 1$$

hence

$$f'(x) = f(x)(\ln x + 1) = x^x(\ln x + 1)$$

2. Let $f(x) = (\sin x)^{\cos x}$. Find $f'(x)$.

Solution: We logarithm and then differentiate both sides of $f(x) = (\sin x)^{\cos x}$. We have

$$f(x) = (\sin x)^{\cos x} \quad\Longrightarrow\quad \ln f(x) = \ln(\sin x)^{\cos x} = \cos x\ln(\sin x)$$

therefore

$$[\ln f(x)]' = [\cos x\ln(\sin x)]'$$

hence

$$\begin{aligned}\frac{1}{f(x)}\cdot f'(x) &= (\cos x)'\ln(\sin x) + \cos x(\ln(\sin x))' \\ &= -\sin x\ln(\sin x) + \cos x\frac{1}{\sin x}(\sin x)' \\ &= -\sin x\ln(\sin x) + \cos x\frac{1}{\sin x}\cos x \\ &= -\sin x\ln(\sin x) + \cos x\cot x\end{aligned}$$

From this it follows that

$$f'(x) = f(x)(-\sin x\ln(\sin x) + \cos x\cot x) = (\sin x)^{\cos x}(-\sin x\ln(\sin x) + \cos x\cot x)$$

3. Let $f(x) = (\sin 2x)^{x^3}$. Find $f'(x)$.

3. Let $f(x)=(\sin 2x)^{x^3}$. Find $f'(x)$.

Solution: We logarithm and then differentiate both sides of $f(x)=(\sin 2x)^{x^3}$. We have

$$f(x)=(\sin 2x)^{x^3} \quad \Longrightarrow \quad \ln f(x)=\ln(\sin 2x)^{x^3}=x^3\ln(\sin 2x)$$

therefore

$$[\ln f(x)]'=\left[x^3\ln(\sin 2x)\right]'$$

hence

$$\begin{aligned}\frac{1}{f(x)}\cdot f'(x) &= (x^3)'\ln(\sin 2x)+x^3(\ln(\sin 2x))'\\ &= 3x^2\ln(\sin 2x)+x^3\frac{1}{\sin 2x}(\sin 2x)'\\ &= 3x^2\ln(\sin 2x)+x^3\frac{1}{\sin 2x}\cos 2x(2x)'\\ &= 3x^2\ln(\sin 2x)+x^3\frac{1}{\sin 2x}\cos 2x\cdot 2\\ &= 3x^2\ln(\sin 2x)+2x^3\cot 2x\end{aligned}$$

From this it follows that

$$f'(x)=f(x)\left(3x^2\ln(\sin 2x)+2x^3\cot 2x\right)=(\sin 2x)^{x^3}(3x^2\ln(\sin 2x)+2x^3\cot 2x)$$

EXAMPLE: Find the derivative of

$$y=\frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}$$

Solution: We have (see Appendix I, Example 5)

$$\ln y=\ln\left(\frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}\right)=\ln x+\frac{1}{2}\ln(x+1)-\frac{1}{3}\ln(x+2)-5\ln(x+3)$$

therefore

$$(\ln y)'=\left(\ln x+\frac{1}{2}\ln(x+1)-\frac{1}{3}\ln(x+2)-5\ln(x+3)\right)'$$

so

$$\frac{1}{y}\cdot y'=\frac{1}{x}+\frac{1}{2}\cdot\frac{1}{x+1}-\frac{1}{3}\cdot\frac{1}{x+2}-5\frac{1}{x+3}=\frac{1}{x}+\frac{1}{2(x+1)}-\frac{1}{3(x+2)}-\frac{5}{x+3}$$

It follows that

$$y'=y\left(\frac{1}{x}+\frac{1}{2(x+1)}-\frac{1}{3(x+2)}-\frac{5}{x+3}\right)=\frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}\left(\frac{1}{x}+\frac{1}{2(x+1)}-\frac{1}{3(x+2)}-\frac{5}{x+3}\right)$$

EXAMPLE: Find the derivative of

$$y=\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$$

EXAMPLE: Find the derivative of

$$y = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$$

Solution: We have (see Appendix I, Example 6)

$$\ln y = \ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right) = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

therefore

$$(\ln y)' = \left(2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\right)'$$

so

$$\frac{1}{y}\cdot y' = \frac{2}{x} + \frac{7}{3(7x-14)} - 4\frac{2x}{1+x^2} = \frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}$$

It follows that

$$y' = y\left(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right) = \frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\left(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right)$$

EXAMPLE: Find the derivative of

$$y = \frac{\sqrt[3]{x^2-8}\sqrt{x^3+1}}{\sqrt{1-x}(x+2)^{-3}(x^6-7x+5)^5}$$

Solution: We have (see Appendix I, Example 7)

$$\ln y = \ln\left(\frac{\sqrt[3]{x^2-8}\sqrt{x^3+1}}{\sqrt{1-x}(x+2)^{-3}(x^6-7x+5)^5}\right)$$

$$= \frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \frac{1}{2}\ln(1-x) + 3\ln(x+2) - 5\ln(x^6-7x+5)$$

therefore

$$(\ln y)' = \left(\frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \frac{1}{2}\ln(1-x) + 3\ln(x+2) - 5\ln(x^6-7x+5)\right)'$$

so

$$\frac{1}{y}\cdot y' = \frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} + \frac{1}{2(1-x)} + \frac{3}{x+2} - \frac{30x^5-35}{x^6-7x+5}$$

It follows that

$$y' = y\left(\frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} + \frac{1}{2(1-x)} + \frac{3}{x+2} - \frac{30x^5-35}{x^6-7x+5}\right)$$

$$= \frac{\sqrt[3]{x^2-8}\sqrt{x^3+1}}{\sqrt{1-x}(x+2)^{-3}(x^6-7x+5)^5}\left(\frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} + \frac{1}{2(1-x)} + \frac{3}{x+2} - \frac{30x^5-35}{x^6-7x+5}\right)$$

Appendix I

1. $\ln(x(x+1)) = \ln x + \ln(x+1)$

2. $\ln\left(\dfrac{x}{x+2}\right) = \ln x - \ln(x+2)$

3. $\ln\left(\dfrac{x(x+1)}{x+2}\right) = \ln\Big(x(x+1)\Big) - \ln(x+2) = \ln x + \ln(x+1) - \ln(x+2)$

4. $$\begin{aligned}\ln\left(\frac{x(x+1)}{(x+2)(x+3)}\right) &= \ln\Big(x(x+1)\Big) - \ln\Big((x+2)(x+3)\Big) \\ &= \Big(\ln x + \ln(x+1)\Big) - \Big(\ln(x+2) + \ln(x+3)\Big) \\ &= \ln x + \ln(x+1) - \ln(x+2) - \ln(x+3)\end{aligned}$$

5. $$\begin{aligned}\ln\left(\frac{x\sqrt{x+1}}{\sqrt[3]{x+2}(x+3)^5}\right) &= \ln\left(\frac{x(x+1)^{1/2}}{(x+2)^{1/3}(x+3)^5}\right) \\ &= \ln\Big(x(x+1)^{1/2}\Big) - \ln\Big((x+2)^{1/3}(x+3)^5\Big) \\ &= \Big(\ln x + \ln(x+1)^{1/2}\Big) - \Big(\ln(x+2)^{1/3} + \ln(x+3)^5\Big) \\ &= \ln x + \ln(x+1)^{1/2} - \ln(x+2)^{1/3} - \ln(x+3)^5 \\ &= \ln x + \frac{1}{2}\ln(x+1) - \frac{1}{3}\ln(x+2) - 5\ln(x+3)\end{aligned}$$

6. $$\begin{aligned}\ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right) = \ln\left(\frac{x^2(7x-14)^{1/3}}{(1+x^2)^4}\right) &= \ln\Big(x^2(7x-14)^{1/3}\Big) - \ln(1+x^2)^4 \\ &= \ln x^2 + \ln(7x-14)^{1/3} - \ln(1+x^2)^4 \\ &= 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\end{aligned}$$

7. $$\begin{aligned}&\ln\left(\frac{\sqrt[3]{x^2-8}\sqrt{x^3+1}}{\sqrt{1-x}(x+2)^{-3}(x^6-7x+5)^5}\right) = \ln\left(\frac{(x^2-8)^{1/3}(x^3+1)^{1/2}}{(1-x)^{1/2}(x+2)^{-3}(x^6-7x+5)^5}\right) \\ &= \ln\Big((x^2-8)^{1/3}(x^3+1)^{1/2}\Big) - \ln\Big((1-x)^{1/2}(x+2)^{-3}(x^6-7x+5)^5\Big) \\ &= \Big(\ln(x^2-8)^{1/3} + \ln(x^3+1)^{1/2}\Big) - \Big(\ln(1-x)^{1/2} + \ln(x+2)^{-3} + \ln(x^6-7x+5)^5\Big) \\ &= \ln(x^2-8)^{1/3} + \ln(x^3+1)^{1/2} - \ln(1-x)^{1/2} - \ln(x+2)^{-3} - \ln(x^6-7x+5)^5 \\ &= \frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \frac{1}{2}\ln(1-x) + 3\ln(x+2) - 5\ln(x^6-7x+5)\end{aligned}$$

Appendix II

The following problem was given in Fall 2013 (Calculus I, Midterm Exam II). None of the 86 test takers got the right answer.

Let $f(x) = \log_2 \left(\dfrac{1}{1+\log_3 x} \right)$, then $f'(x)$ is

(A) $\dfrac{1}{x(1+\log_3 x)\ln 2 \ln 3}$

(B) $\dfrac{1}{x(1-\log_3 x)\ln 2 \ln 3}$

(C) $-\dfrac{1}{x(1-\log_3 x)\ln 2 \ln 3}$

(D) $-\dfrac{1}{x \ln 2 \ln(3x)}$

(E) None of the above

See the next page for the solution.

Let $f(x) = \log_2\left(\dfrac{1}{1+\log_3 x}\right)$, then $f'(x)$ is

Ⓐ $\dfrac{1}{x(1+\log_3 x)\ln 2 \ln 3}$

Ⓑ $\dfrac{1}{x(1-\log_3 x)\ln 2 \ln 3}$

Ⓒ $-\dfrac{1}{x(1-\log_3 x)\ln 2 \ln 3}$

Ⓓ $-\dfrac{1}{x\ln 2 \ln(3x)}$ ⟵ Correct

Ⓔ None of the above

Solution: We first note that

$$\log_2\left(\frac{1}{1+\log_3 x}\right) = \log_2\left[(1+\log_3 x)^{-1}\right] = (-1)\log_2(1+\log_3 x) = -\log_2(1+\log_3 x)$$

or

$$\log_2\left(\frac{1}{1+\log_3 x}\right) = \log_2 1 - \log_2(1+\log_3 x) = 0 - \log_2(1+\log_3 x) = -\log_2(1+\log_3 x)$$

Therefore

$$\left[\log_2\left(\frac{1}{1+\log_3 x}\right)\right]' = [-\log_2(1+\log_3 x)]' = -[\log_2(1+\log_3 x)]'$$

$$= -\frac{1}{(1+\log_3 x)\ln 2}\cdot(1+\log_3 x)'$$

$$= -\frac{1}{(1+\log_3 x)\ln 2}\cdot\frac{1}{x\ln 3}$$

$$= -\frac{1}{(1+\log_3 x)x\ln 2\ln 3}$$

$$= -\frac{1}{\left(1+\dfrac{\ln x}{\ln 3}\right)x\ln 2\ln 3}$$

$$= -\frac{1}{\left(1\cdot\ln 3+\dfrac{\ln x}{\ln 3}\cdot\ln 3\right)x\ln 2}$$

$$= -\frac{1}{(\ln 3+\ln x)\,x\ln 2}$$

$$= -\frac{1}{\ln(3x)\,x\ln 2}$$