Inverse Functions and Logarithms

DEFINITION: A function \( f \) is called a one-to-one function if it never takes on the same value twice; that is,

\[ f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2 \]

HORIZONTAL LINE TEST: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLES:

1. Functions \( x, x^3, x^5, 1/x, \) etc. are one-to-one, since if \( x_1 \neq x_2 \), then

\[ x_1 \neq x_2, \quad x_1^3 \neq x_2^3, \quad x_1^5 \neq x_2^5, \quad \frac{1}{x_1} \neq \frac{1}{x_2} \]

2. Functions \( x^2, x^4, \sin x, \) etc. are not one-to-one, since

\[ (-1)^2 = 1^2, \quad (-1)^4 = 1^4, \quad \sin 0 = \sin \pi \]

DEFINITION: Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). Then its inverse function \( f^{-1} \) has domain \( B \) and range \( A \) and is defined by

\[ f^{-1}(y) = x \iff f(x) = y \]  \((*)\)

for any \( y \) in \( B \).

So, we can reformulate \((*)\) as

\[ (f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \text{ for every } x \text{ in the domain of } f \]

\[ (f \circ f^{-1})(x) = f(f^{-1}(x)) = x \text{ for every } x \text{ in the domain of } f^{-1} \]

IMPORTANT: Do not confuse \( f^{-1} \) with \( \frac{1}{f} \).
(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \text{ for every } x \text{ in the domain of } f

(f \circ f^{-1})(x) = f(f^{-1}(x)) = x \text{ for every } x \text{ in the domain of } f^{-1}

EXAMPLES:

1. Let \( f(x) = x^3 \), then \( f^{-1}(x) = \sqrt[3]{x} \), since
   \[ f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x \quad \text{and} \quad f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x \]

2. Let \( f(x) = x^3 + 1 \), then \( f^{-1}(x) = \sqrt[3]{x - 1} \), since
   \[ f^{-1}(f(x)) = f^{-1}(x^3 + 1) = \sqrt[3]{x^3 + 1} - 1 = x \quad \text{and} \quad f(f^{-1}(x)) = f(\sqrt[3]{x - 1}) = (\sqrt[3]{x - 1})^3 + 1 = x \]

3. Let \( f(x) = 2x \), then \( f^{-1}(x) = \frac{1}{2}x \), since
   \[ f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x \]

4. Let \( f(x) = x \), then \( f^{-1}(x) = x \), since
   \[ f^{-1}(f(x)) = f^{-1}(x) = x \quad \text{and} \quad f(f^{-1}(x)) = f(x) = x \]

5. Let \( f(x) = 7x + 2 \), then \( f^{-1}(x) = \frac{x - 2}{7} \), since
   \[ f^{-1}(f(x)) = f^{-1}(7x + 2) = \frac{(7x + 2) - 2}{7} = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{x - 2}{7}\right) = 7\left(\frac{x - 2}{7}\right) + 2 = x \]

Solution: We have:

Step 1: Replace \( f(x) \) by \( y \):
   \[ y = 7x + 2 \]

Step 2: Solve for \( x \):
   \[ y = 7x + 2 \quad \implies \quad y - 2 = 7x \quad \implies \quad \frac{y - 2}{7} = x \]

Step 3: Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):
   \[ f^{-1}(x) = \frac{x - 2}{7} \]

6. Let \( f(x) = (3x - 2)^5 + 2 \). Find \( f^{-1}(x) \).
6. Let \( f(x) = (3x - 2)^5 + 2 \). Find \( f^{-1}(x) \).

Solution: We have:

**Step 1: Replace \( f(x) \) by \( y \):**

\[ y = (3x - 2)^5 + 2 \]

**Step 2: Solve for \( x \):**

\[ y = (3x - 2)^5 + 2 \implies y - 2 = (3x - 2)^5 \implies \sqrt[5]{y - 2} = x - 2 \implies \sqrt[5]{y - 2} + 2 = 3x \]

therefore

\[ x = \frac{\sqrt[5]{y - 2} + 2}{3} \]

**Step 3: Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):**

\[ f^{-1}(x) = \frac{\sqrt[5]{x - 2} + 2}{3} \]

7. Let \( f(x) = \frac{3x - 5}{4 - 2x} \). Find \( f^{-1}(x) \).

8. Let \( f(x) = \sqrt{x} \). Find \( f^{-1}(x) \).
7. Let \( f(x) = \frac{3x - 5}{4 - 2x} \), then \( f^{-1}(x) = \frac{4x + 5}{3 + 2x} \).

Solution: We have:

**Step 1:** Replace \( f(x) \) by \( y \):

\[
y = \frac{3x - 5}{4 - 2x}
\]

**Step 2:** Solve for \( x \):

\[
y = \frac{3x - 5}{4 - 2x} \implies y(4 - 2x) = 3x - 5 \implies 4y - 2xy = 3x - 5 \implies 4y + 5 = 3x + 2xy
\]

therefore

\[
4y + 5 = x(3 + 2y) \implies \frac{4y + 5}{3 + 2y} = x
\]

**Step 3:** Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):

\[
f^{-1}(x) = \frac{4x + 5}{3 + 2x}
\]

8. Let \( f(x) = \sqrt{x} \), then \( f^{-1}(x) = x^2, \ x \geq 0 \).

**IMPORTANT:**

\[
\text{domain of } f^{-1} = \text{range of } f
\]

\[
\text{range of } f^{-1} = \text{domain of } f
\]

9. Let \( f(x) = \sqrt{3x - 2} \), then \( f^{-1}(x) = \frac{1}{3}(x^2 + 2), \ x \geq 0 \) (see Appendix, page 9).

10. Let \( f(x) = \sqrt{x - 1} \), then \( f^{-1}(x) = x^4 + 1, \ x \geq 0 \) (see Appendix, page 9).

11. Let \( f(x) = \sqrt{x + 5} + 1 \), then \( f^{-1}(x) = (x - 1)^2 - 5, \ x \geq 1 \) (see Appendix, page 10).

12. Let \( f(x) = \sqrt[4]{2x - 7} + 5 \). Find \( f^{-1}(x) \).
12. Let \( f(x) = \sqrt[4]{2x-7} + 5 \), then \( f^{-1}(x) = \frac{(x-5)^4 + 7}{2} \), \( x \geq 5 \) (see Appendix, page 10).

13. The function \( f(x) = x^2 \) is not invertible, since it is not a one-to-one function.

REMARK: Similarly, \( x^4, \ x^{10}, \ \sin x, \ \cos x, \ \text{etc.} \)

are not invertable functions.

14. The function \( f(x) = (x + 1)^2 \) is not invertible.

15. Let \( f(x) = x^2, \ x \geq 0 \), then \( f^{-1}(x) = \sqrt{x}, \ x \geq 0 \).

16. Let \( f(x) = x^2, \ x \geq 2 \), then \( f^{-1}(x) = \sqrt{x}, \ x \geq 4 \).

17. Let \( f(x) = x^2, \ x < -3 \), then \( f^{-1}(x) = -\sqrt{x}, \ x > 9 \).

18. The function \( f(x) = x^2, \ x > -1 \) is not invertible.

19. Let \( f(x) = (x + 1)^2, \ x > 3 \). Find \( f^{-1}(x) \).

20. Let \( f(x) = (1 + 2x)^2, \ x \leq -1 \). Find \( f^{-1}(x) \).
19. Let \( f(x) = (x + 1)^2, x > 3 \), then \( f^{-1}(x) = \sqrt{x} - 1, x > 16 \) (see Appendix, page 11).

20. Let \( f(x) = (1 + 2x)^2, x \leq -1 \), then \( f^{-1}(x) = -\sqrt{x} + 1 = \frac{1}{2}, x \geq 1 \) (see Appendix, page 11).

**THEOREM:** If \( f \) has an inverse function \( f^{-1} \), then the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) are reflections of one another about the line \( y = x \); that is, each is the mirror image of the other with respect to that line.

![Graph of functions](image)

**THEOREM:** If \( f \) is a one-to-one continuous function defined on an interval, then its inverse function \( f^{-1} \) is also continuous.

**THEOREM (Differentiability of Inverse Functions):** If \( f \) is a one-to-one differentiable function with inverse function \( f^{-1} \) and \( f'(f^{-1}(a)) \neq 0 \), then the inverse function is differentiable at \( a \) and

\[
(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}
\]

**EXAMPLE:** If \( f(x) = x^5 + x + 2 \), find \( (f^{-1})'(4) \).

Solution 1: We have \( (f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} \). Since \( f(1) = 4 \), it follows that \( f^{-1}(4) = 1 \). Hence

\[
(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(1)}
\]

But \( f'(x) = 5x^4 + 1 \), therefore \( f'(1) = 5 \cdot 1^4 + 1 = 6 \). This yields

\[
(f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{6}
\]

Solution 2: One can see that \( y = f^{-1}(x) \) satisfies the equation \( x = y^5 + y + 2 \). To find \( y' \) we differentiate both sides:

\[
x' = (y^5 + y + 2)' \implies 1 = 5y^4 \cdot y' + y' \implies 1 = y'(5y^4 + 1) \implies y' = \frac{1}{5y^4 + 1}
\]

Note that if \( x = 4 \), then \( y = 1 \) (solution of \( 4 = y^5 + y + 2 \)). Therefore

\[
(f^{-1})'(4) = y'(4) = \frac{1}{5 \cdot 1^4 + 1} = \frac{1}{6}
\]
Logarithmic Functions

If \( a > 0 \) and \( a \neq 1 \), the exponential function \( f(x) = a^x \) is either increasing or decreasing and so it is one-to one by the Horizontal Line Test. It therefore has an inverse function \( f^{-1}(x) \), which is called the logarithmic function with base \( a \) and is denoted by \( \log_a x \). We have

\[
\log_a x = y \iff a^y = x
\]

BASIC PROPERTIES: \( f(x) = \log_a x \) is a continuous function with domain \((0, \infty)\) and range \((-\infty, \infty)\). Moreover,

\[
\log_a(a^x) = x \text{ for every } x \in \mathbb{R}, \quad a^\log_a x = x \text{ for every } x > 0
\]

REMARK: It immediately follows from property 1 that

\[
\log_a a = 1, \quad \log_a 1 = 0
\]

LAWS OF LOGARITHMS: If \( x \) and \( y \) are positive numbers, then

1. \( \log_a(xy) = \log_a x + \log_a y \).
2. \( \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y \).
3. \( \log_a (x^r) = r \log_a x \) where \( r \) is any real number.

EXAMPLES:
1. Use the laws of logarithms to evaluate \( \log_3 270 - \log_3 10 \).
Solution: We have

\[
\log_3 270 - \log_3 10 = \log_3 \left( \frac{270}{10} \right) = \log_3 27 = \log_3 3^3 = 3 \log_3 3 = 3 \cdot 1 = 3
\]

2. Use the laws of logarithms to evaluate \( \log_2 12 + \log_2 3 - \log_2 9 \).
2. Use the laws of logarithms to evaluate $\log_2 12 + \log_2 3 - \log_2 9$.

Solution: We have

$$\log_2 12 + \log_2 3 - \log_2 9 = \log_2 (12 \cdot 3) - \log_2 9 = \log_2 \left( \frac{12 \cdot 3}{9} \right) = \log_2 4 = \log_2 2^2 = 2 \log_2 2 = 2 \cdot 1 = 2$$

**BASIC CALCULUS PROPERTIES:**

1. If $a > 1$, then $\lim_{x \to \infty} \log_a x = \infty$ and $\lim_{x \to 0^+} \log_a x = -\infty$.

2. If $0 < a < 1$, then $\lim_{x \to \infty} \log_a x = -\infty$ and $\lim_{x \to 0^+} \log_a x = \infty$.

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**Natural Logarithms**

**DEFINITION:** The logarithm with base $e$ is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x$$

**BASIC PROPERTIES:**

1. $\ln(e^x) = x$ for every $x \in \mathbb{R}$.

2. $e^{\ln x} = x$ for every $x > 0$.

**REMARK:** It immediately follows from property 1 that

$$\ln e = 1$$

**IMPORTANT FORMULA:** For any positive $a$ and $b$ ($a, b \neq 1$) we have

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular, if $a = e$, then

$$\log_e x = \frac{\ln x}{\ln b}$$
9. Let \( f(x) = \sqrt{3x - 2} \), then \( f^{-1}(x) = \frac{1}{3}(x^2 + 2), \ x \geq 0 \).

Solution: We have:

**Step 1:** Replace \( f(x) \) by \( y \):

\[
y = \sqrt{3x - 2}
\]

**Step 2:** Solve for \( x \):

\[
y = \sqrt{3x - 2} \quad \Rightarrow \quad y^2 = 3x - 2 \quad \Rightarrow \quad y^2 + 2 = 3x
\]

therefore

\[
x = \frac{1}{3}(y^2 + 2)
\]

**Step 3:** Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):

\[
f^{-1}(x) = \frac{1}{3}(x^2 + 2)
\]

Finally, since the range of \( f \) is all nonnegative numbers, it follows that the domain of \( f^{-1} \) is \( x \geq 0 \).

10. Let \( f(x) = \sqrt[4]{x - 1} \), then \( f^{-1}(x) = x^4 + 1, \ x \geq 0 \).

Solution: We have:

**Step 1:** Replace \( f(x) \) by \( y \):

\[
y = \sqrt[4]{x - 1}
\]

**Step 2:** Solve for \( x \):

\[
y = \sqrt[4]{x - 1} \quad \Rightarrow \quad y^4 = x - 1
\]

therefore

\[
x = y^4 + 1
\]

**Step 3:** Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):

\[
f^{-1}(x) = x^4 + 1
\]

Finally, since the range of \( f \) is all nonnegative numbers, it follows that the domain of \( f^{-1} \) is \( x \geq 0 \).
11. Let \( f(x) = \sqrt{x+5} + 1 \), then \( f^{-1}(x) = (x - 1)^2 - 5, \ x \geq 1. \)

Solution: We have:

Step 1: Replace \( f(x) \) by \( y \):
\[
y = \sqrt{x+5} + 1
\]

Step 2: Solve for \( x \):
\[
y = \sqrt{x+5} + 1 \implies y - 1 = \sqrt{x+5} \implies (y - 1)^2 = x + 5
\]
therefore
\[
x = (y - 1)^2 - 5
\]

Step 3: Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):
\[
f^{-1}(x) = (x - 1)^2 - 5
\]

Finally, since the range of \( f \) is all numbers \( \geq 1 \), it follows that the domain of \( f^{-1} \) is \( x \geq 1. \)

12. Let \( f(x) = \sqrt{2x - 7} + 5 \), then \( f^{-1}(x) = \frac{(x - 5)^4 + 7}{2}, \ x \geq 5. \)

Solution: We have:

Step 1: Replace \( f(x) \) by \( y \):
\[
y = \sqrt{2x - 7} + 5
\]

Step 2: Solve for \( x \):
\[
y = \sqrt{2x - 7} + 5 \implies y - 5 = \sqrt{2x - 7} \implies (y - 5)^4 = 2x - 7 \implies (y - 5)^4 + 7 = 2x
\]
therefore
\[
x = \frac{(y - 5)^4 + 7}{2}
\]

Step 3: Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):
\[
f^{-1}(x) = \frac{(x - 5)^4 + 7}{2}
\]

Finally, since the range of \( f \) is all numbers \( \geq 5 \), it follows that the domain of \( f^{-1} \) is \( x \geq 5. \)
19. Let \( f(x) = (x + 1)^2, x > 3, \) then \( f^{-1}(x) = \sqrt{x} - 1, x > 16. \)

Solution: We have:

**Step 1:** Replace \( f(x) \) by \( y \): 
\[
y = (x + 1)^2
\]

**Step 2:** Solve for \( x \):
\[
y = (x + 1)^2 \quad \text{\( x \) is positive} \quad \sqrt{y} = x + 1
\]
therefore
\[
x = \sqrt{y} - 1
\]

**Step 3:** Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):
\[
f^{-1}(x) = \sqrt{x} - 1
\]

To find the domain of \( f^{-1} \) we note that the range of \( f \) is all numbers \( > 16 \). Indeed, since \( x > 3 \), we have
\[
f(x) = (x + 1)^2 > (3 + 1)^2 = 4^2 = 16
\]
From this it follows that the domain of \( f^{-1} \) is \( x > 16 \).

20. Let \( f(x) = (1 + 2x)^2, x \leq -1, \) then \( f^{-1}(x) = -\sqrt{x + 1}, x \geq 1. \)

Solution: We have:

**Step 1:** Replace \( f(x) \) by \( y \): 
\[
y = (1 + 2x)^2
\]

**Step 2:** Solve for \( x \):
\[
y = (1 + 2x)^2 \quad \text{\( x \leq -1 \)} \quad -\sqrt{y} = 1 + 2x \quad \Rightarrow \quad -\sqrt{y} - 1 = 2x
\]
therefore
\[
x = -\frac{\sqrt{y} + 1}{2}
\]

**Step 3:** Replace \( x \) by \( f^{-1}(x) \) and \( y \) by \( x \):
\[
f^{-1}(x) = -\frac{\sqrt{x} + 1}{2}
\]

To find the domain of \( f^{-1} \) we note that the range of \( f \) is all numbers \( \geq 1 \). Indeed, since \( x \leq -1 \), we have
\[
f(x) = (1 + 2x)^2 \geq (1 + 2 \cdot (-1))^2 = (1 - 2)^2 = (-1)^2 = 1
\]
From this it follows that the domain of \( f^{-1} \) is \( x \geq 1. \)