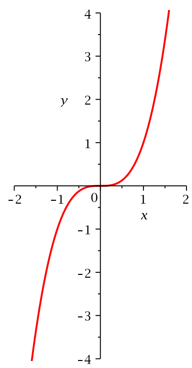
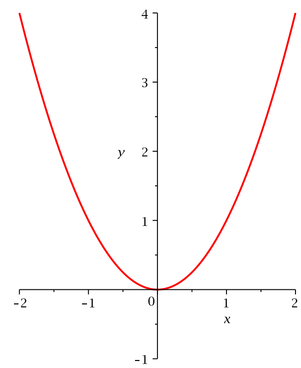
Inverse Functions and Logarithms

DEFINITION: A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2$$



one-to-one



not one-to-one

HORIZONTAL LINE TEST: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

EXAMPLES:

1. Functions $x,\ x^3,\ x^5,\ 1/x$, etc. are one-to-one, since if $x_1 \neq x_2$, then

$$x_1 \neq x_2, \quad x_1^3 \neq x_2^3, \quad x_1^5 \neq x_2^5, \quad \frac{1}{x_1} \neq \frac{1}{x_2}$$

2. Functions $x^2,\ x^4,\ \sin x$, etc. are not one-to-one, since

$$(-1)^2 = 1^2, \quad (-1)^4 = 1^4, \quad \sin 0 = \sin \pi$$

DEFINITION: Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Longleftrightarrow \quad f(x) = y \qquad (*)$$

for any y in B.

So, we can reformulate $(*)$ as

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$	for every x in the domain of f
$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$	for every x in the domain of f^{-1}

<u>IMPORTANT</u>: Do not confuse f^{-1} with $\dfrac{1}{f}$.

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x \quad \text{for every } x \text{ in the domain of } f$$
$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x \quad \text{for every } x \text{ in the domain of } f^{-1}$$

EXAMPLES:

1. Let $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$, since

$$f^{-1}(f(x)) = f^{-1}(x^3) = \sqrt[3]{x^3} = x \quad \text{and} \quad f(f^{-1}(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

2. Let $f(x) = x^3 + 1$, then $f^{-1}(x) = \sqrt[3]{x-1}$, since

$$f^{-1}(f(x)) = f^{-1}(x^3+1) = \sqrt[3]{(x^3+1)-1} = x \quad \text{and} \quad f(f^{-1}(x)) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3+1 = x$$

3. Let $f(x) = 2x$, then $f^{-1}(x) = \frac{1}{2}x$, since

$$f^{-1}(f(x)) = f^{-1}(2x) = \frac{1}{2}(2x) = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{1}{2}x\right) = 2\left(\frac{1}{2}x\right) = x$$

4. Let $f(x) = x$, then $f^{-1}(x) = x$, since

$$f^{-1}(f(x)) = f^{-1}(x) = x \quad \text{and} \quad f(f^{-1}(x)) = f(x) = x$$

5. Let $f(x) = 7x + 2$, then $f^{-1}(x) = \frac{x-2}{7}$, since

$$f^{-1}(f(x)) = f^{-1}(7x+2) = \frac{(7x+2)-2}{7} = x \quad \text{and} \quad f(f^{-1}(x)) = f\left(\frac{x-2}{7}\right) = 7\left(\frac{x-2}{7}\right)+2 = x$$

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = 7x + 2$$

Step 2: Solve for x:

$$y = 7x + 2 \quad \Longrightarrow \quad y - 2 = 7x \quad \Longrightarrow \quad \frac{y-2}{7} = x$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{x-2}{7}$$

6. Let $f(x) = (3x-2)^5 + 2$. Find $f^{-1}(x)$.

6. Let $f(x)=(3x-2)^5+2$. Find $f^{-1}(x)$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y=(3x-2)^5+2$$

Step 2: Solve for x:

$$y=(3x-2)^5+2 \quad\Longrightarrow\quad y-2=(3x-2)^5 \quad\Longrightarrow\quad \sqrt[5]{y-2}=3x-2 \quad\Longrightarrow\quad \sqrt[5]{y-2}+2=3x$$

therefore

$$x=\frac{\sqrt[5]{y-2}+2}{3}$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x)=\frac{\sqrt[5]{x-2}+2}{3}$$

7. Let $f(x)=\dfrac{3x-5}{4-2x}$. Find $f^{-1}(x)$.

8. Let $f(x)=\sqrt{x}$. Find $f^{-1}(x)$.

7. Let $f(x) = \dfrac{3x-5}{4-2x}$, then $f^{-1}(x) = \dfrac{4x+5}{3+2x}$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = \frac{3x-5}{4-2x}$$

Step 2: Solve for x:

$$y = \frac{3x-5}{4-2x} \quad \Longrightarrow \quad y(4-2x) = 3x-5 \quad \Longrightarrow \quad 4y - 2xy = 3x - 5 \quad \Longrightarrow \quad 4y + 5 = 3x + 2xy$$

therefore

$$4y + 5 = x(3+2y) \quad \Longrightarrow \quad \frac{4y+5}{3+2y} = x$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{4x+5}{3+2x}$$

8. Let $f(x) = \sqrt{x}$, then $f^{-1}(x) = x^2,\ x \geq 0$.

IMPORTANT:

$$\boxed{\begin{array}{l} \text{domain of } f^{-1} = \text{ range of } f \\ \text{range of } f^{-1} = \text{ domain of } f \end{array}}$$

9. Let $f(x) = \sqrt{3x-2}$, then $f^{-1}(x) = \dfrac{1}{3}(x^2+2),\ x \geq 0$ (see Appendix, page 9).

10. Let $f(x) = \sqrt[4]{x-1}$, then $f^{-1}(x) = x^4 + 1,\ x \geq 0$ (see Appendix, page 9).

11. Let $f(x) = \sqrt{x+5} + 1$, then $f^{-1}(x) = (x-1)^2 - 5,\ x \geq 1$ (see Appendix, page 10).

12. Let $f(x) = \sqrt[4]{2x-7} + 5$. Find $f^{-1}(x)$.

$$\boxed{\begin{array}{l}\text{domain of } f^{-1} = \text{ range of } f \\ \text{range of } f^{-1} = \text{ domain of } f\end{array}}$$

12. Let $f(x) = \sqrt[4]{2x-7}+5$, then $f^{-1}(x) = \dfrac{(x-5)^4+7}{2}$, $x \geq 5$ (see Appendix, page 10).

13. The function $f(x) = x^2$ is not invertible, since it is not a one-to-one function.

REMARK: Similarly,

$$x^4, \quad x^{10}, \quad \sin x, \quad \cos x, \quad \text{etc.}$$

are not invertable functions.

14. The function $f(x) = (x+1)^2$ is not invertible.

15. Let $f(x) = x^2, x \geq 0$, then $f^{-1}(x) = \sqrt{x}, x \geq 0$.

16. Let $f(x) = x^2, x \geq 2$, then $f^{-1}(x) = \sqrt{x}, x \geq 4$.

17. Let $f(x) = x^2, x < -3$, then $f^{-1}(x) = -\sqrt{x}, x > 9$.

18. The function $f(x) = x^2, x > -1$ is not invertible.

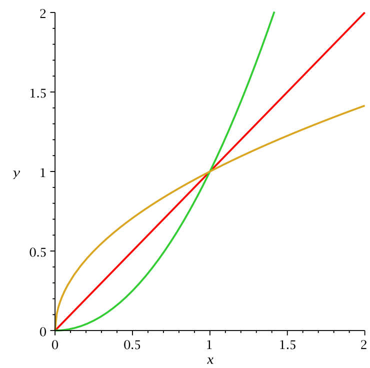
19. Let $f(x) = (x+1)^2, x > 3$. Find $f^{-1}(x)$.

20. Let $f(x) = (1+2x)^2, x \leq -1$. Find $f^{-1}(x)$.

19. Let $f(x)=(x+1)^2, x>3$, then $f^{-1}(x)=\sqrt{x}-1, x>16$ (see Appendix, page 11).

20. Let $f(x)=(1+2x)^2, x\le -1$, then $f^{-1}(x)=-\dfrac{\sqrt{x}+1}{2}, x\ge 1$ (see Appendix, page 11).

THEOREM: If f has an inverse function f^{-1}, then the graphs of $y=f(x)$ and $y=f^{-1}(x)$ are reflections of one another about the line $y=x$; that is, each is the mirror image of the other with respect to that line.



THEOREM: If f is a one-to-one continuous function defined on an interval, then its inverse function f^{-1} is also continuous.

THEOREM **(Differentiability of Inverse Functions):** If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a))\neq 0$, then the inverse function is differentiable at a and

$$\boxed{(f^{-1})'(a)=\frac{1}{f'(f^{-1}(a))}}$$

EXAMPLE: If $f(x)=x^5+x+2$, find $(f^{-1})'(4)$.

Solution 1: We have $(f^{-1})'(4)=\dfrac{1}{f'(f^{-1}(4))}$. Since $f(1)=4$, it follows that $f^{-1}(4)=1$. Hence

$$(f^{-1})'(4)=\frac{1}{f'(f^{-1}(4))}=\frac{1}{f'(1)}$$

But $f'(x)=5x^4+1$, therefore $f'(1)=5\cdot 1^4+1=6$. This yields

$$(f^{-1})'(4)=\frac{1}{f'(1)}=\frac{1}{6}$$

Solution 2: One can see that $y=f^{-1}(x)$ satisfies the equation $x=y^5+y+2$. To find y' we differentiate both sides:

$$x'=(y^5+y+2)' \quad\Longrightarrow\quad 1=5y^4\cdot y'+y' \quad\Longrightarrow\quad 1=y'(5y^4+1) \quad\Longrightarrow\quad y'=\frac{1}{5y^4+1}$$

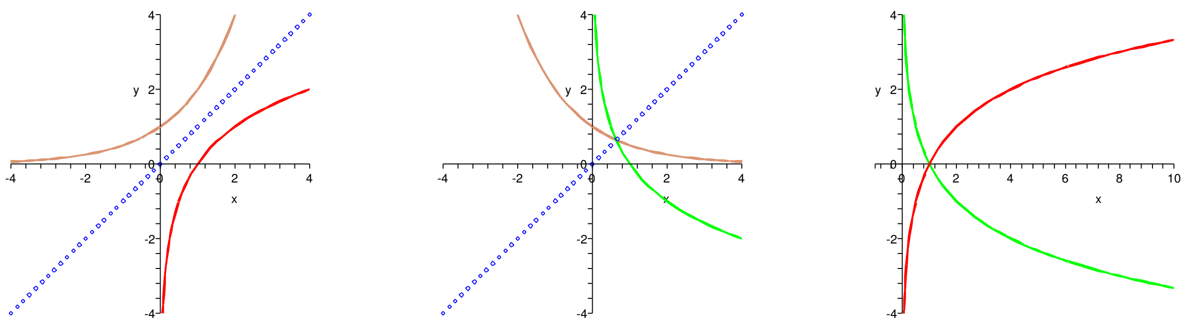
Note that if $x=4$, then $y=1$ (solution of $4=y^5+y+2$). Therefore

$$(f^{-1})'(4)=y'(4)=\frac{1}{5\cdot 1^4+1}=\frac{1}{6}$$

Logarithmic Functions

If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to one by the Horizontal Line Test. It therefore has an inverse function $f^{-1}(x)$, which is called the **logarithmic function with base** a and is denoted by $\log_a x$. We have

$$\boxed{\log_a x = y \quad \Longleftrightarrow \quad a^y = x}$$



BASIC PROPERTIES: $f(x) = \log_a x$ is a continuous function with domain $(0, \infty)$ and range $(-\infty, \infty)$. Moreover,

$$\boxed{\log_a(a^x) = x \text{ for every } x \in \mathbb{R}, \quad a^{\log_a x} = x \text{ for every } x > 0}$$

REMARK: It immediately follows from property 1 that

$$\boxed{\log_a a = 1, \quad \log_a 1 = 0}$$

LAWS OF LOGARITHMS: If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$.

2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

3. $\log_a(x^r) = r\log_a x$ where r is any real number.

EXAMPLES:

1. Use the laws of logarithms to evaluate $\log_3 270 - \log_3 10$.

Solution: We have

$$\log_3 270 - \log_3 10 = \log_3\left(\frac{270}{10}\right) = \log_3 27 = \log_3 3^3 = 3\log_3 3 = 3 \cdot 1 = 3$$

2. Use the laws of logarithms to evaluate $\log_2 12 + \log_2 3 - \log_2 9$.

2. Use the laws of logarithms to evaluate $\log_2 12 + \log_2 3 - \log_2 9$.

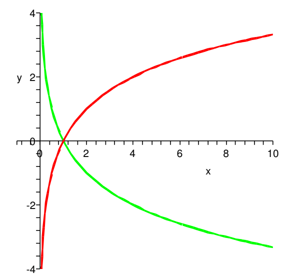
Solution: We have

$$\log_2 12 + \log_2 3 - \log_2 9 = \log_2(12\cdot 3) - \log_2 9 = \log_2\left(\frac{12\cdot 3}{9}\right) = \log_2 4 = \log_2 2^2$$
$$= 2\log_2 2 = 2\cdot 1 = 2$$

BASIC CALCULUS PROPERTIES:

1. If $a > 1$, then $\lim\limits_{x\to\infty} \log_a x = \infty$ and $\lim\limits_{x\to 0^+} \log_a x = -\infty$.

2. If $0 < a < 1$, then $\lim\limits_{x\to\infty} \log_a x = -\infty$ and $\lim\limits_{x\to 0^+} \log_a x = \infty$.



Natural Logarithms

DEFINITION: The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\boxed{\log_e x = \ln x}$$

BASIC PROPERTIES:

1. $\ln(e^x) = x$ for every $x \in \mathbb{R}$.

2. $e^{\ln x} = x$ for every $x > 0$.

REMARK: It immediately follows from property 1 that

$$\ln e = 1$$

IMPORTANT FORMULA: For any positive a and b $(a, b \neq 1)$ we have

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

In particular, if $a = e$, then

$$\boxed{\log_b x = \frac{\ln x}{\ln b}}$$

Appendix

9. Let $f(x) = \sqrt{3x-2}$, then $f^{-1}(x) = \dfrac{1}{3}(x^2+2),\ x \geq 0$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = \sqrt{3x-2}$$

Step 2: Solve for x:

$$y = \sqrt{3x-2} \quad \Longrightarrow \quad y^2 = 3x-2 \quad \Longrightarrow \quad y^2+2 = 3x$$

therefore

$$x = \frac{1}{3}(y^2+2)$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{1}{3}(x^2+2)$$

Finally, since the range of f is all nonnegative numbers, it follows that the domain of f^{-1} is $x \geq 0$.

10. Let $f(x) = \sqrt[4]{x-1}$, then $f^{-1}(x) = x^4+1,\ x \geq 0$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = \sqrt[4]{x-1}$$

Step 2: Solve for x:

$$y = \sqrt[4]{x-1} \quad \Longrightarrow \quad y^4 = x-1$$

therefore

$$x = y^4+1$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = x^4+1$$

Finally, since the range of f is all nonnegative numbers, it follows that the domain of f^{-1} is $x \geq 0$.

11. Let $f(x) = \sqrt{x+5}+1$, then $f^{-1}(x) = (x-1)^2 - 5, \ x \geq 1$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = \sqrt{x+5}+1$$

Step 2: Solve for x:

$$y = \sqrt{x+5}+1 \quad \Longrightarrow \quad y-1 = \sqrt{x+5} \quad \Longrightarrow \quad (y-1)^2 = x+5$$

therefore

$$x = (y-1)^2 - 5$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = (x-1)^2 - 5$$

Finally, since the range of f is all numbers ≥ 1, it follows that the domain of f^{-1} is $x \geq 1$.

12. Let $f(x) = \sqrt[4]{2x-7}+5$, then $f^{-1}(x) = \dfrac{(x-5)^4+7}{2}, \ x \geq 5$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = \sqrt[4]{2x-7}+5$$

Step 2: Solve for x:

$$y = \sqrt[4]{2x-7}+5 \quad \Longrightarrow \quad y-5 = \sqrt[4]{2x-7} \quad \Longrightarrow \quad (y-5)^4 = 2x-7 \quad \Longrightarrow \quad (y-5)^4+7 = 2x$$

therefore

$$x = \frac{(y-5)^4+7}{2}$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{(x-5)^4+7}{2}$$

Finally, since the range of f is all numbers ≥ 5, it follows that the domain of f^{-1} is $x \geq 5$.

19. Let $f(x) = (x+1)^2, x > 3$, then $f^{-1}(x) = \sqrt{x} - 1, x > 16$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = (x+1)^2$$

Step 2: Solve for x:

$$y = (x+1)^2 \quad \overset{x \text{ is positive}}{\Longrightarrow} \quad \sqrt{y} = x + 1$$

therefore

$$x = \sqrt{y} - 1$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \sqrt{x} - 1$$

To find the domain of f^{-1} we note that the range of f is all numbers > 16. Indeed, since $x > 3$, we have

$$f(x) = (x+1)^2 > (3+1)^2 = 4^2 = 16$$

From this it follows that the domain of f^{-1} is $x > 16$.

20. Let $f(x) = (1+2x)^2, x \leq -1$, then $f^{-1}(x) = -\dfrac{\sqrt{x}+1}{2}, x \geq 1$.

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = (1+2x)^2$$

Step 2: Solve for x:

$$y = (1+2x)^2 \quad \overset{x \leq -1}{\Longrightarrow} \quad -\sqrt{y} = 1 + 2x \quad \Longrightarrow \quad -\sqrt{y} - 1 = 2x$$

therefore

$$x = -\frac{\sqrt{y}+1}{2}$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = -\frac{\sqrt{x}+1}{2}$$

To find the domain of f^{-1} we note that the range of f is all numbers ≥ 1. Indeed, since $x \leq -1$, we have

$$f(x) = (1+2x)^2 \geq (1 + 2 \cdot (-1))^2 = (1-2)^2 = (-1)^2 = 1$$

From this it follows that the domain of f^{-1} is $x \geq 1$.