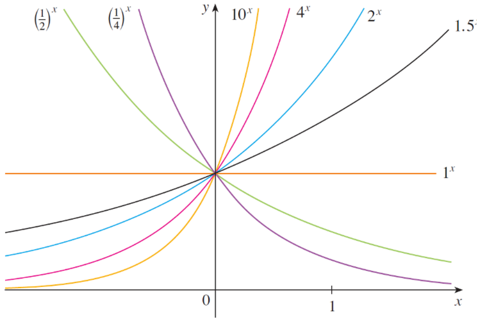
Exponential Functions

DEFINITION: An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant.

x	$\left(\frac{1}{10}\right)^x$
-3	$\left(\frac{1}{10}\right)^{-3} = 10^3 = 1000$
-2	$\left(\frac{1}{10}\right)^{-2} = 10^2 = 100$
-1	$\left(\frac{1}{10}\right)^{-1} = 10^1 = 10$
0	$\left(\frac{1}{10}\right)^{0} = 1$
1	$\left(\frac{1}{10}\right)^{1} = \frac{1}{10} = 0.1$
2	$\left(\frac{1}{10}\right)^{2} = \frac{1}{10^2} = 0.01$
3	$\left(\frac{1}{10}\right)^{3} = \frac{1}{10^3} = 0.001$



x	10^x
-3	$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$
-2	$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$
-1	$10^{-1} = \frac{1}{10} = 0.1$
0	$10^0 = 1$
1	$10^1 = 10$
2	$10^2 = 100$
3	$10^3 = 1000$

BASIC ALGEBRAIC PROPERTIES:

1. $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$ if n is a positive integer.
2. $a^0 = 1$.
3. $a^{-n} = \dfrac{1}{a^n}$.
4. $a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

THEOREM: If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. In particular, $a^x > 0$ for all x. If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^{x+y} = a^x a^y$ 2. $a^{x-y} = \dfrac{a^x}{a^y}$ 3. $(a^x)^y = a^{xy}$ 4. $(ab)^x = a^x b^x$

BASIC CALCULUS PROPERTIES:

1. If $a > 1$, then $\lim\limits_{x\to\infty} a^x = \infty$ and $\lim\limits_{x\to-\infty} a^x = 0$.

2. If $0 < a < 1$, then $\lim\limits_{x\to\infty} a^x = 0$ and $\lim\limits_{x\to-\infty} a^x = \infty$.

EXAMPLES:

1. Find $\lim\limits_{x\to-\infty}(5^x + 2^x + 4)$.

Solution: Since $\lim\limits_{x\to-\infty} 5^x = 0$ and $\lim\limits_{x\to-\infty} 2^x = 0$, we have

$$\lim_{x\to-\infty}(5^x + 2^x + 4) = \lim_{x\to-\infty} 5^x + \lim_{x\to-\infty} 2^x + \lim_{x\to-\infty} 4 = 0 + 0 + 4 = 4$$

2. Find $\lim\limits_{x\to\infty}(0.1^x + 0.7^x)$ and $\lim\limits_{x\to\infty}(0.1^x + 0.7^{-x})$.

2. Find $\lim\limits_{x\to\infty}(0.1^x+0.7^x)$ and $\lim\limits_{x\to\infty}(0.1^x+0.7^{-x})$.

Solution: Since $\lim\limits_{x\to\infty}0.1^x=0$ and $\lim\limits_{x\to\infty}0.7^x=0$, we have

$$\lim_{x\to\infty}(0.1^x+0.7^x)=\lim_{x\to\infty}0.1^x+\lim_{x\to\infty}0.7^x=0+0=0$$

Similarly, since $\lim\limits_{x\to\infty}0.1^x=0$ and $\lim\limits_{x\to\infty}0.7^{-x}=\infty$, we have

$$\lim_{x\to\infty}(0.1^x+0.7^{-x})=\infty$$

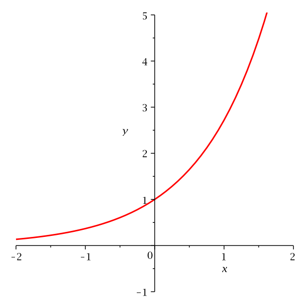
The Number e and the Natural Exponential Function

It is known that

$$\lim_{x\to 0}(1+x)^{1/x}=\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x=2.7182818284590452353602874713526624977572470936...$$

We denote this number by e.

x	$\left(1+\frac{1}{x}\right)^x$	Value
1	2^1	2
10	1.1^{10}	2.593742460
100	1.01^{100}	2.704813829
1000	1.001^{1000}	2.716923932
10000	1.0001^{10000}	2.718145927



DEFINITION: The **natural exponential function** is $f(x)=e^x$.

PROPERTIES OF THE NATURAL EXPONENTIAL FUNCTION: The exponential function $f(x)=e^x$ is a continuous function with domain $\mathbb{R}$ and range $(0,\infty)$. Thus $e^x>0$ for all x. Also

$$\lim_{x\to-\infty}e^x=0 \qquad \lim_{x\to\infty}e^x=\infty$$

So the x-axis is a horizontal asymptote of $f(x)=e^x$.

EXAMPLES:

1. Find $\lim\limits_{x\to\infty}e^{-x^3-x+5}$.

Solution: Put $t=-x^3-x+5$. We know that $t\to-\infty$ as $x\to\infty$. Therefore

$$\lim_{x\to\infty}e^{-x^3-x+5}=\lim_{t\to-\infty}e^t=0$$

2. Find $\lim\limits_{x\to 3^-}e^{2/(x-3)}$ and $\lim\limits_{x\to 3^+}e^{2/(x-3)}$.

2. Find $\lim\limits_{x\to 3^-} e^{2/(x-3)}$ and $\lim\limits_{x\to 3^+} e^{2/(x-3)}$.

Solution: Put $t = 2/(x-3)$. We know that $t\to -\infty$ as $x\to 3^-$ and $t\to\infty$ as $x\to 3^+$. Therefore

$$\lim_{x\to 3^-} e^{2/(x-3)} = \lim_{t\to-\infty} e^t = 0$$

and

$$\lim_{x\to 3^+} e^{2/(x-3)} = \lim_{t\to\infty} e^t = \infty$$

3. Find $\lim\limits_{x\to\infty} \dfrac{49^x - 7^x + 8}{5\cdot 7^{2x} - 1}$.

Solution: Put $t = 7^x$. We know that $t\to\infty$ as $x\to\infty$. Therefore

$$\lim_{x\to\infty} \frac{49^x - 7^x + 8}{5\cdot 7^{2x} - 1} = \lim_{x\to\infty} \frac{(7^x)^2 - 7^x + 8}{5\cdot (7^x)^2 - 1} = \lim_{t\to\infty} \frac{t^2 - t + 8}{5t^2 - 1} = \lim_{t\to\infty} \frac{\frac{t^2-t+8}{t^2}}{\frac{5t^2-1}{t^2}} = \lim_{t\to\infty} \frac{\frac{t^2}{t^2} - \frac{t}{t^2} + \frac{8}{t^2}}{\frac{5t^2}{t^2} - \frac{1}{t^2}}$$

$$= \lim_{t\to\infty} \frac{1 - \frac{1}{t} + \frac{8}{t^2}}{5 - \frac{1}{t^2}} = \frac{1 - 0 + 0}{5 - 0} = \frac{1}{5}$$

4. Find $\lim\limits_{x\to\infty} \dfrac{e^x}{\sqrt{e^{2x}+1}}$.

5. Find $\lim\limits_{x\to\infty} e^{-2x}\sin x$.

4. Find $\lim\limits_{x\to\infty} \dfrac{e^x}{\sqrt{e^{2x}+1}}$.

Solution: Put $t = e^x$. We know that $t \to \infty$ as $x \to \infty$. Therefore

$$\lim_{x\to\infty} \frac{e^x}{\sqrt{e^{2x}+1}} = \lim_{t\to\infty} \frac{t}{\sqrt{t^2+1}} = \lim_{t\to\infty} \frac{\frac{t}{\sqrt{t^2}}}{\frac{\sqrt{t^2+1}}{\sqrt{t^2}}} = \lim_{t\to\infty} \frac{\frac{t}{t}}{\sqrt{\frac{t^2+1}{t^2}}}$$

$$= \lim_{t\to\infty} \frac{1}{\sqrt{\frac{t^2}{t^2}+\frac{1}{t^2}}} = \lim_{t\to\infty} \frac{1}{\sqrt{1+\frac{1}{t^2}}} = \frac{1}{\sqrt{1+0}} = 1$$

5. Find $\lim\limits_{x\to\infty} e^{-2x}\sin x$.

Solution: Put $t = -2x$. We know that $t \to -\infty$ as $x \to \infty$. Therefore

$$\lim_{x\to\infty} e^{-2x}\sin x = \lim_{t\to-\infty} e^t \sin\left(-\frac{t}{2}\right)$$

Note that

$$-1 \le \sin\left(-\frac{t}{2}\right) \le 1$$

Multiplying all three parts of this inequality by e^t, we get

$$-e^t \le e^t \sin\left(-\frac{t}{2}\right) \le e^t$$

Since

$$\lim_{t\to-\infty} \left(-e^t\right) = \lim_{t\to-\infty} e^t = 0$$

it follows that

$$\lim_{t\to-\infty} e^t \sin\left(-\frac{t}{2}\right) = 0$$

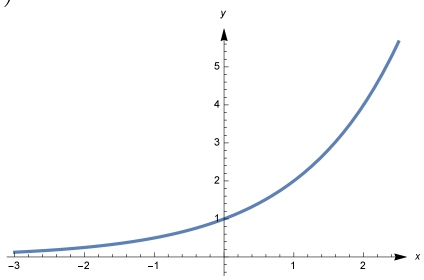
by the Squeeze Theorem. Thus

$$\lim_{x\to\infty} e^{-2x}\sin x = 0$$

Appendix

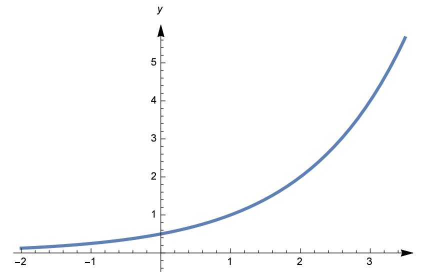
EXAMPLE: Graph the following functions:

(a) $f(x) = 2^{x-1}$



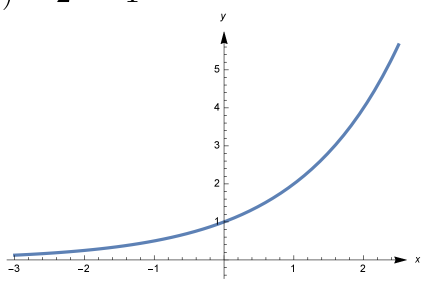
$y = 2^x$

$\Longrightarrow$



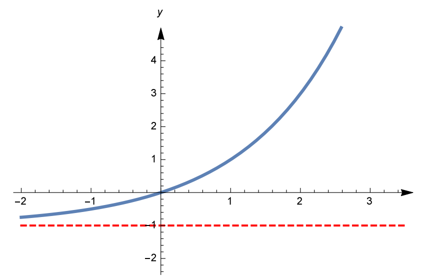
$f(x) = 2^{x-1}$ (horizontal shift)

(b) $g(x) = 2^x - 1$



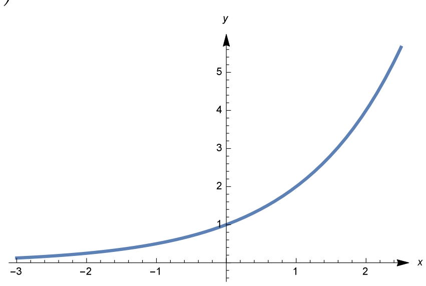
$y = 2^x$

$\Longrightarrow$



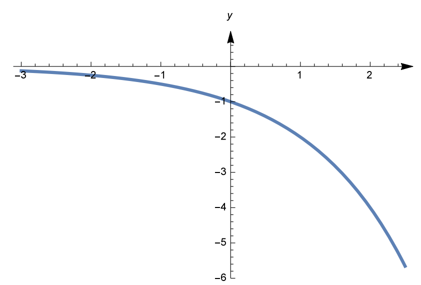
$g(x) = 2^x - 1$ (vertical shift)

(c) $h(x) = -2^x$



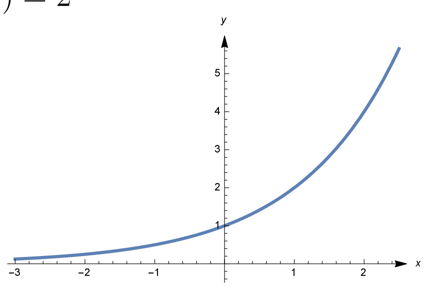
$y = 2^x$

$\Longrightarrow$



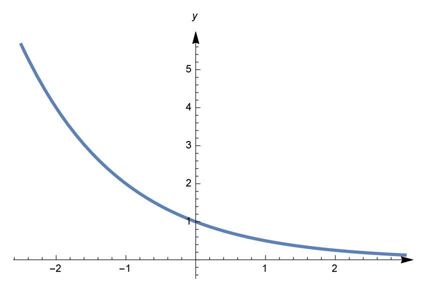
$h(x) = -2^x$ (reflection)

(d) $p(x) = 2^{-x}$



$y = 2^x$

$\Longrightarrow$



$p(x) = 2^{-x}$ (reflection)

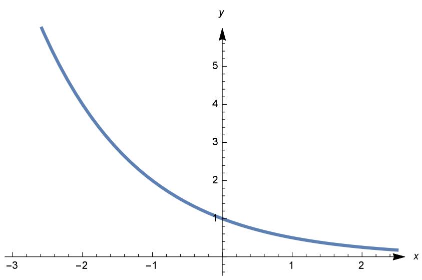
REMARK: An other way to graph $p(x)$ is to rewrite it as $p(x) = \dfrac{1}{2^x} = \left(\dfrac{1}{2}\right)^x$, which gives the same result by the Figure on page 1.

EXAMPLE: Graph $f(x) = -2^{-x}$.

Solution: Note that

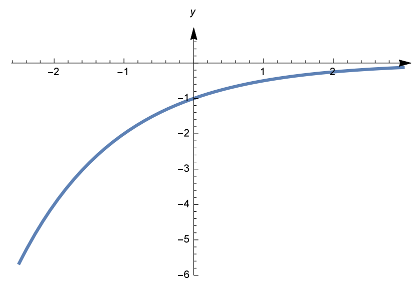
$$f(x) = -2^{-x} = -\frac{1}{2^x} = -\left(\frac{1}{2}\right)^x$$

Therefore



$$y = \left(\frac{1}{2}\right)^x$$

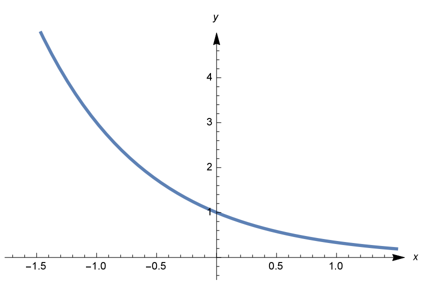
$\Longrightarrow$



$$f(x) = -\left(\frac{1}{2}\right)^x \text{ (reflection)}$$

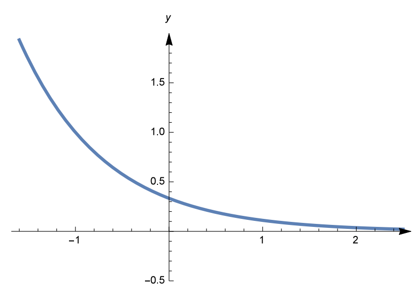
EXAMPLE: Graph $f(x) = \left(\frac{1}{3}\right)^{x+1} + 2$.

Solution: We have



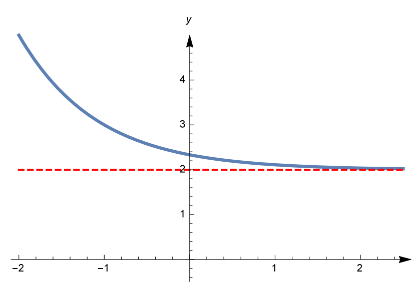
$$y = \left(\frac{1}{3}\right)^x$$

$\Longrightarrow$



$$y = \left(\frac{1}{3}\right)^{x+1} \text{ (horizontal shift)}$$

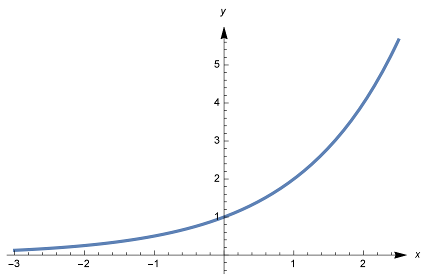
$\Downarrow$



$$f(x) = \left(\frac{1}{3}\right)^{x+1} + 2 \text{ (vertical shift)}$$

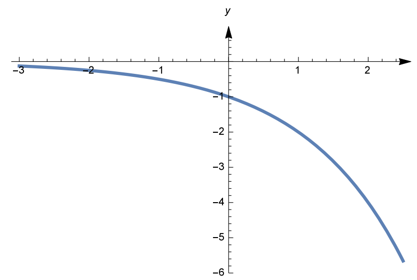
EXAMPLE: Graph $f(x) = -e^{1-x} + 3$.

Solution: We have



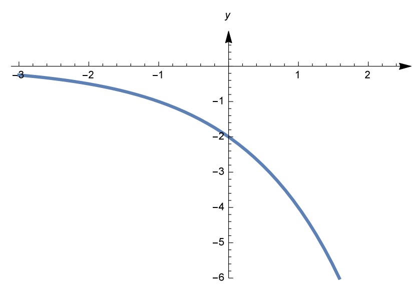
$y = 2^x$

$\Longrightarrow$



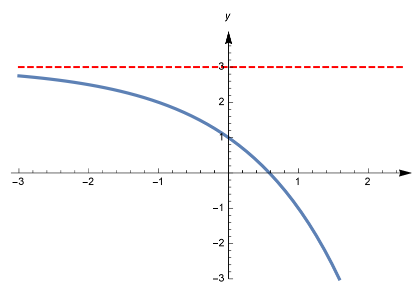
$y = -2^x$ (reflection)

$\Downarrow$



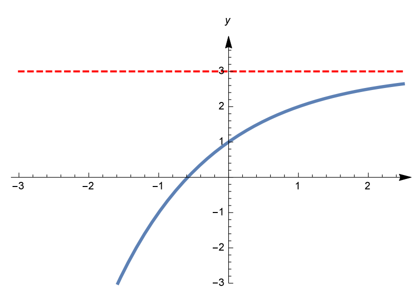
$y = -2^{x+1}$ (horizontal shift)

$\Downarrow$



$y = -2^{x+1} + 3$ (vertical shift)

$\Downarrow$



$f(x) = -2^{-x+1} + 3$ (reflection)