Implicit Differentiation

Some functions can be described by expressing one variable explicitly in terms of another variable — for example,

\[ y = x^2, \quad y = \sqrt{\frac{1 - x}{1 + x^3}}, \quad y = \tan 2x \]

or, in general, \( y = f(x) \). Some functions, however, are defined implicitly by a relation between \( x \) and \( y \) such that

\[ x^2 + y^2 = 1, \quad x^3 + y^3 = 4xy, \quad (x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \]

In some cases it is possible to solve such an equation for \( x \) or for \( y \), but sometimes it is impossible. One of the main goals of Section 2.6 is to show how to find derivatives of implicitly defined functions.

EXAMPLE: Find an equation of the tangent line to \( y = x^2 \) at the point \((2, 4)\).

Solution: The graph of the curve \( y = x^2 \) is a parabola. Clearly,

\[ y' = (x^2)' = 2x \]

To find an equation of the tangent line to \( y = x^2 \) at the point \((2, 4)\) we note that

\[ m = y'(2) = 2 \cdot 2 = 4 \]

therefore

\[ y - y_0 = m(x - x_0) \implies y - 4 = 4(x - 2) \implies y = 4x - 4 \]

EXAMPLE: Find equations of the tangent lines to \( x = y^2 \) at the points \((4, 2)\) and \((4, -2)\).
EXAMPLE: Find equations of the tangent lines to \( x = y^2 \) at the points \((4, 2)\) and \((4, -2)\).

Solution 1: The graph of the curve \( x = y^2 \) is a parabola. We have

\[
x = y^2 \quad \Rightarrow \quad y = \begin{cases} \sqrt{x} & \text{if } y \geq 0 \\ -\sqrt{x} & \text{if } y < 0 \end{cases}
\]

Clearly, if \( y = \sqrt{x} \), then

\[
y' = (x^{1/2})' = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad (1)
\]

Similarly, if \( y = -\sqrt{x} \), then

\[
y' = (-x^{1/2})' = -\frac{1}{2}x^{1/2-1} = -\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad (2)
\]

To find an equation of the tangent line to \( y^2 = x \) at the point \((4, 2)\) we note that by (1) we have

\[
m_1 = y'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}
\]

therefore

\[
y - y_1 = m_1(x - x_1) \quad \Rightarrow \quad y - 2 = \frac{1}{4}(x - 4) \quad \Rightarrow \quad y = \frac{1}{4}x + 1
\]

Similarly, to find an equation of the tangent line to \( y^2 = x \) at the point \((4, -2)\) we note that

\[
m_2 = y'(4) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4}
\]

therefore

\[
y - y_2 = m_2(x - x_2) \quad \Rightarrow \quad y - (-2) = -\frac{1}{4}(x - 4) \quad \Rightarrow \quad y = -\frac{1}{4}x - 1
\]

Solution 2: To find equations of the tangent lines to \( x = y^2 \) at the points \((4, 2)\) and \((4, -2)\) we first find \( \frac{dy}{dx} \) by differentiating both sides of \( x = y^2 \):

\[
x = y^2 \quad \Rightarrow \quad x' = (y^2)' \quad \Rightarrow \quad 1 = 2y \cdot y' \quad \Rightarrow \quad y' = \frac{1}{2y}
\]

It follows that

\[
m = y'(4) = \begin{cases} \frac{1}{2 \cdot 2} & \text{if } y = 2 \\ \frac{1}{2 \cdot (-2)} & \text{if } y = -2 \end{cases}
\]

therefore

\[
y - y_1 = m_1(x - x_1) \quad \Rightarrow \quad y - 2 = \frac{1}{4}(x - 4) \quad \Rightarrow \quad y = \frac{1}{4}x + 1 \quad \text{at } (4, 2)
\]

and

\[
y - y_2 = m_2(x - x_2) \quad \Rightarrow \quad y - (-2) = -\frac{1}{4}(x - 4) \quad \Rightarrow \quad y = -\frac{1}{4}x - 1 \quad \text{at } (4, -2)
\]
EXAMPLE: If \( x^2 + y^2 = 5 \), find \( \frac{dy}{dx} \). Then find an equation of the tangent line to \( x^2 + y^2 = 5 \) at the point (2, 1).

Solution: The graph of the curve \( x^2 + y^2 = 5 \) is a circle:

To find \( \frac{dy}{dx} \) we differentiate both sides:

\[
\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(5) \implies (x^2)' + (y^2)' = 0 \implies 2x \cdot x' + 2y \cdot y' = 0
\]

hence

\[
2x \cdot 1 + 2y \cdot y' = 0 \implies 2x + 2y \cdot y' = 0 \implies 2y \cdot y' = -2x \implies y' = -\frac{2x}{2y} = -\frac{x}{y}
\]

To find an equation of the tangent line to \( x^2 + y^2 = 5 \) at the point (2, 1) we note that

\[ m = y'(2) = -\frac{2}{1} = -2 \]

therefore

\[ y - y_0 = m(x - x_0) \implies y - 1 = -2 \cdot (x - 2) \implies y = -2x + 5 \]

EXAMPLE: If \( 2x^2 + 3y^2 = 5 \), find \( \frac{dy}{dx} \). Then find an equation of the tangent line to \( 2x^2 + 3y^2 = 5 \) at the point (1, 1).
EXAMPLE: If $2x^2 + 3y^2 = 5$, find $\frac{dy}{dx}$. Then find an equation of the tangent line to $2x^2 + 3y^2 = 5$ at the point (1, 1).

Solution: The graph of the curve $2x^2 + 3y^2 = 5$ is an ellipse:

To find $\frac{dy}{dx}$ we differentiate both sides:

$2x^2 + 3y^2 = 5 \implies (2x^2 + 3y^2)' = 5' \implies 2(x^2)' + 3(y^2)' = 0 \implies 2(2x \cdot x') + 3(2y \cdot y') = 0$

hence

$2(2x \cdot 1) + 3(2y \cdot y') = 0 \implies 4x + 6y \cdot y' = 0 \implies 6y \cdot y' = -4x \implies y' = -\frac{4x}{6y} = -\frac{2x}{3y}$

To find an equation of the tangent line to $2x^2 + 3y^2 = 5$ at the point (1, 1) we note that

$m = y'(1) = -\frac{2 \cdot 1}{3 \cdot 1} = -\frac{2}{3}$

therefore

$y - y_0 = m(x - x_0) \implies y - 1 = -\frac{2}{3}(x - 1) \implies y = -\frac{2}{3}x + \frac{5}{3}$

EXAMPLE: If $x^3 + y^3 = 4xy$, find $\frac{dy}{dx}$. Then find an equation of the tangent line to $x^3 + y^3 = 4xy$ at the point (2, 2). At what point in the first quadrant is the tangent line horizontal?
EXAMPLE: If \( x^3 + y^3 = 4xy \), find \( \frac{dy}{dx} \). Then find an equation of the tangent line to \( x^3 + y^3 = 4xy \) at the point \((2, 2)\). At what point in the first quadrant is the tangent line horizontal?

Solution: The graph of the curve \( x^3 + y^3 = 4xy \) is the folium of Descartes:

(a) To find \( \frac{dy}{dx} \) we differentiate both sides:

\[
x^3 + y^3 = 4xy \quad \Rightarrow \quad (x^3 + y^3)' = (4xy)' \quad \Rightarrow \quad (x^3)' + (y^3)' = (4xy)'
\]

Since \((4xy)' = 4(xy)' = 4(x'y + xy') = 4(1 \cdot y + xy') = 4(y + xy')\), this gives us

\[
3x^2 \cdot x' + 3y^2 \cdot y' = 4(y + xy') \quad \Rightarrow \quad 3x^2 + 3y^2 \cdot y' = 4y + 4xy' \quad \Rightarrow \quad 3y^2 \cdot y' - 4xy' = 4y - 3x^2
\]

hence

\[
y'(3y^2 - 4) = 4y - 3x^2 \quad \Rightarrow \quad y' = \frac{4y - 3x^2}{3y^2 - 4x}
\]

(b) To find an equation of the tangent line to \( x^3 + y^3 = 4xy \) at the point \((2, 2)\) we note that

\[
m = y'(2) = \frac{4 \cdot 2 - 3 \cdot 2^2}{3 \cdot 2^2 - 4 \cdot 2} = -1
\]

therefore

\[
y - y_0 = m(x - x_0) \quad \Rightarrow \quad y - 2 = -1 \cdot (x - 2) \quad \Rightarrow \quad y = -x + 4
\]
(c) The tangent line is horizontal if \( y' = 0 \). Using the expression for \( y' \) from part (a), we see that \( y' = 0 \) when \( 4y - 3x^2 = 0 \) (provided that \( 3y^2 - 4x \neq 0 \)). We have

\[
4y - 3x^2 = 0 \implies y = \frac{3}{4}x^2 \quad \quad \quad \quad x^3 + \left( \frac{3}{4}x^2 \right)^3 = 4x \left( \frac{3}{4}x^2 \right) \implies x^3 + \frac{27}{64}x^6 = 3x^3
\]

which gives

\[
\frac{27}{64}x^6 = 2x^3
\]

Dividing both sides by 2, we get \( \frac{27}{128}x^6 = x^3 \). Since \( x \neq 0 \) in the first quadrant, we can divide both sides by \( x^3 \) which implies

\[
\frac{27}{128}x^3 = 1 \implies x^3 = \frac{128}{27}
\]

hence

\[
x = \sqrt[3]{\frac{128}{27}} = \left\{ \frac{\sqrt[3]{128}}{\sqrt[3]{27}} = \frac{\sqrt{64}\cdot 2}{\sqrt[3]{27}} = \frac{\sqrt{64}\sqrt{2}}{\sqrt[3]{27}} = \frac{4\sqrt{2}}{3} \right\} = \frac{4}{3}\sqrt{2} \approx 1.6798947
\]

Plugging in this into \( y = \frac{3}{4}x^2 \), we get

\[
y = \frac{3}{4} \left( \frac{4}{3}\sqrt{2} \right)^2 = \frac{4}{3} \sqrt{4} \approx 2.1165347
\]

Finally, one can check that \( 3y^2 - 4x \neq 0 \) at \( \left( \frac{4}{3}\sqrt{2}, \frac{4}{3}\sqrt{4} \right) \). Thus the tangent line is horizontal at \( \left( \frac{4}{3}\sqrt{2}, \frac{4}{3}\sqrt{4} \right) \), which is approximately \( (1.6798947, 2.1165347) \). Looking at the figure, we see that our answer is reasonable.

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EXAMPLE: If \((x^2+y^2-2x)^2 = 4(x^2+y^2)\), find an equation of the tangent line to \((x^2+y^2-2x)^2 = 4(x^2+y^2)\) at the point \((0, 2)\).
EXAMPLE: If \((x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)\), find an equation of the tangent line to \((x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)\) at the point \((0, 2)\).

Solution: The graph of the curve \((x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)\) is the cardioid:

We differentiate both sides of \((x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)\):

\[
[(x^2 + y^2 - 2x)^2]' = [4(x^2 + y^2)]' \implies 2(x^2 + y^2 - 2x)^2 \cdot (x^2 + y^2 - 2x)' = 4(x^2 + y^2)' \]

hence

\[
2(x^2 + y^2 - 2x)(2x + 2y \cdot y' - 2) = 4(2x + 2y \cdot y') \tag{3}
\]

To find slope of the tangent line to \((x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)\) at the point \((0, 2)\) we replace \(x\) by 0 and \(y\) by 2 in (3):

\[
2(0^2 + 2^2 - 2 \cdot 2)(2 \cdot 0 + 2 \cdot 2 \cdot y' - 2) = 4(2 \cdot 0 + 2 \cdot 2 \cdot y') \implies 8(4y' - 2) = 4(4y')
\]

hence

\[
32y' - 16 = 16y' \implies 32y' - 16y' = 16 \implies 16y' = 16 \implies y' = \frac{16}{16} = 1
\]

so \(m = 1\), therefore

\[
y - y_0 = m(x - x_0) \implies y - 2 = 1 \cdot (x - 0) \implies y = x + 2
\]

REMARK: If we first find \(\frac{dy}{dx}\) for any \(x\) and then find the slope of the tangent line to \((x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)\) at the point \((0, 2)\), the computations will be more complicated (see Appendix, page 10).
EXAMPLE: If \((x^2 + y^2)^3 = 4x^2y^2\), find \(\frac{dy}{dx}\).

Solution: The graph of the curve \((x^2 + y^2)^3 = 4x^2y^2\) is a rose curve:

\[
\begin{align*}
(x^2 + y^2)^3 &= 4x^2y^2 \\
\Rightarrow \quad [(x^2 + y^2)^3]' &= (4x^2y^2)'
\end{align*}
\]

hence

\[
3(x^2 + y^2)^{3-1} \cdot (x^2 + y^2)' = 4(x^2y^2)'
\]

Note that

\[
(x^2 + y^2)' = (x^2)' + (y^2)' = 2x + 2y \cdot y' = 2(x + y \cdot y')
\]

and

\[
(x^2y^2)' = (x^2)'y^2 + x^2(y^2)' = 2xy^2 + x^2 \cdot 2y \cdot y' = 2(xy^2 + x^2y \cdot y')
\]

Substituting (5) and (6) into (4), we obtain

\[
3(x^2 + y^2)^2 \cdot 2(x + y \cdot y') = 4 \cdot 2(xy^2 + x^2y \cdot y')
\]

\[
3(x^2 + y^2)(x + y \cdot y') = 4(xy^2 + x^2y \cdot y')
\]

\[
3(x^4 + 2x^2y^2 + y^4)(x + y \cdot y') = 4xy^2 + 4x^2y \cdot y'
\]

We now expand the parentheses and solve this equation for \(y'\):

\[
3x^5 + 3x^4y \cdot y' + 6x^3y^2 + 6x^2y^3 \cdot y' + 3xy^4 + 3y^5 \cdot y' = 4xy^2 + 4x^2y \cdot y'
\]

so

\[
3x^4y \cdot y' + 6x^2y^3 \cdot y' + 3y^5 \cdot y' - 4x^2y \cdot y' = -3x^5 - 6x^3y^2 - 3xy^4 + 4xy^2
\]

hence

\[
y'(3x^4y + 6x^2y^3 + 3y^5 - 4x^2y) = -3x^5 - 6x^3y^2 - 3xy^4 + 4xy^2
\]

therefore

\[
y' = \frac{-3x^5 - 6x^3y^2 - 3xy^4 + 4xy^2}{3x^4y + 6x^2y^3 + 3y^5 - 4x^2y} = \frac{3x^5 + 6x^3y^2 + 3y^4x - 4y^2x}{3y^5 + 6y^3x^2 + 3x^4y - 4x^2y}
\]
EXAMPLE: Find $y'$ if $\sin(x + y) = y^2 \cos x$.

Solution: We have

$$\sin(x + y) = y^2 \cos x \implies (\sin(x + y))' = (y^2 \cos x)'$$

therefore

$$\cos(x + y) \cdot (x + y)' = (y^2)' \cos x + y^2(\cos x)'$$

hence

$$\cos(x + y) \cdot (1 + y') = 2y \cdot y' \cos x - y^2 \sin x$$

We now solve this equation for $y'$:

$$\cos(x + y) + \cos(x + y) \cdot y' = 2y \cdot y' \cos x - y^2 \sin x$$

so

$$\cos(x + y) \cdot y' - 2y \cdot y' \cos x = -\cos(x + y) - y^2 \sin x$$

therefore

$$y'(\cos(x + y) - 2y \cos x) = -\cos(x + y) - y^2 \sin x$$

Dividing both sides by $\cos(x + y) - 2y \cos x$, we get

$$y' = \frac{-\cos(x + y) - y^2 \sin x}{\cos(x + y) - 2y \cos x} = \frac{\cos(x + y) + y^2 \sin x}{2y \cos x - \cos(x + y)}$$
EXAMPLE: If \((x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)\), find \(\frac{dy}{dx}\).

Solution: To find \(\frac{dy}{dx}\) we differentiate both sides:
\[
(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2) \implies [(x^2 + y^2 - 2x)^2]' = [4(x^2 + y^2)]'
\]

hence
\[
2(x^2 + y^2 - 2x)^{2-1} \cdot (x^2 + y^2 - 2x)' = 4(x^2 + y^2)'
\]  
(7)

Note that
\[
(x^2 + y^2 - 2x)' = (x^2)' + (y^2)' - (2x)' = 2x + 2y \cdot y' - 2 = 2(x + y \cdot y' - 1)
\]  
(8)

and
\[
(x^2 + y^2)' = (x^2)' + (y^2)' = 2x + 2y \cdot y'
\]  
(9)

Substituting (8) and (9) into (7), we obtain
\[
2(x^2 + y^2 - 2x)(x + y \cdot y' - 1) = 2x + 2y \cdot y'
\]

We now expand the parentheses and solve this equation for \(y'\):
\[
x^3 + x^2y \cdot y' - x^2 + xy^2 + y^3 \cdot y' - y^2 - 2x^2 - 2xy \cdot y' + 2x = 2x + 2y \cdot y'
\]

so
\[
x^2y \cdot y' + y^3 \cdot y' - 2xy \cdot y' - 2y \cdot y' = -x^3 - xy^2 + 3x^2 + y^2
\]

hence
\[
y'(x^2y + y^3 - 2xy - 2y) = -x^3 - xy^2 + 3x^2 + y^2
\]

therefore
\[
y' = \frac{-x^3 - xy^2 + 3x^2 + y^2}{x^2y + y^3 - 2xy - 2y} = \frac{x^3 + xy^2 - 3x^2 - y^2}{y^3 + x^2y - 2xy - 2y}
\]
EXAMPLE: Sketch the curve $r = 1 + \sin \theta$, $0 \leq \theta \leq 2\pi$ (cardioid).

Solution: We have

\[
\begin{align*}
\text{r = 1 + sin(\theta), } & \text{theta = } \pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 2\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 3\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 4\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 5\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 6\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 7\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 8\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 9\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 10\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 11\pi/6 \\
\text{r = 1 + sin(\theta), } & \text{theta = } 12\pi/6
\end{align*}
\]
EXAMPLE: Sketch the curve $r = 1 - \cos \theta$, $0 \leq \theta \leq 2\pi$ (cardioid).

Solution: We have