Implicit Differentiation

Some functions can be described by expressing one variable explicitly in terms of another variable — for example,

$$y = x^2, \quad y = \sqrt{\frac{1-x}{1+x^3}}, \quad y = \tan 2x$$

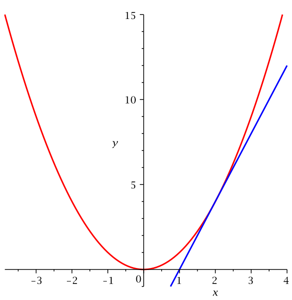
or, in general, $y = f(x)$. Some functions, however, are defined implicitly by a relation between x and y such that

$$x^2 + y^2 = 1, \quad x^3 + y^3 = 4xy, \quad (x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$

In some cases it is possible to solve such an equation for x or for y, but sometimes it is impossible. One of the main goals of Section 2.6 is to show how to find derivatives of implicitly defined functions.

EXAMPLE: Find an equation of the tangent line to $y = x^2$ at the point $(2, 4)$.

Solution: The graph of the curve $y = x^2$ is a parabola. Clearly,



$$y' = (x^2)' = 2x$$

To find an equation of the tangent line to $y = x^2$ at the point $(2, 4)$ we note that

$$m = y'(2) = 2 \cdot 2 = 4$$

therefore

$$y - y_0 = m(x - x_0) \quad \Longrightarrow \quad y - 4 = 4(x - 2) \quad \Longrightarrow \quad y = 4x - 4$$

EXAMPLE: Find equations of the tangent lines to $x = y^2$ at the points $(4, 2)$ and $(4, -2)$.

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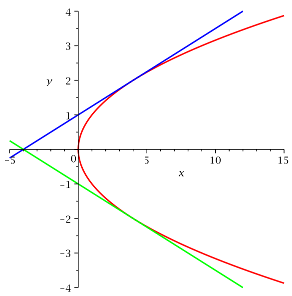
Solution 1: The graph of the curve $x = y^2$ is a parabola. We have



$$x = y^2 \quad\Longrightarrow\quad y = \begin{cases} \sqrt{x} & \text{if } y \geq 0 \\ -\sqrt{x} & \text{if } y < 0 \end{cases}$$

Clearly, if $y = \sqrt{x}$, then

$$y' = (x^{1/2})' = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \tag{1}$$

Similarly, if $y = -\sqrt{x}$, then

$$y' = (-x^{1/2})' = -\frac{1}{2}x^{1/2-1} = -\frac{1}{2}x^{-1/2} = -\frac{1}{2\sqrt{x}} \tag{2}$$

To find an equation of the tangent line to $y^2 = x$ at the point $(4, 2)$ we note that by (1) we have

$$m_1 = y'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

therefore

$$y - y_1 = m_1(x - x_1) \quad\Longrightarrow\quad y - 2 = \frac{1}{4}(x - 4) \quad\Longrightarrow\quad y = \frac{1}{4}x + 1$$

Similarly, to find an equation of the tangent line to $y^2 = x$ at the point $(4, -2)$ we note that by (2) we have

$$m_2 = y'(4) = -\frac{1}{2\sqrt{4}} = -\frac{1}{4}$$

therefore

$$y - y_2 = m_2(x - x_2) \quad\Longrightarrow\quad y - (-2) = -\frac{1}{4}(x - 4) \quad\Longrightarrow\quad y = -\frac{1}{4}x - 1$$

Solution 2: To find equations of the tangent lines to $x = y^2$ at the points $(4, 2)$ and $(4, -2)$ we first find $\dfrac{dy}{dx}$ by differentiating both sides of $x = y^2$:

$$x = y^2 \quad\Longrightarrow\quad x' = (y^2)' \quad\Longrightarrow\quad 1 = 2y \cdot y' \quad\Longrightarrow\quad y' = \frac{1}{2y}$$

It follows that

$$m = y'(4) = \begin{cases} \dfrac{1}{2 \cdot 2} & \text{if } y = 2 \\[2ex] \dfrac{1}{2 \cdot (-2)} & \text{if } y = -2 \end{cases}$$

therefore

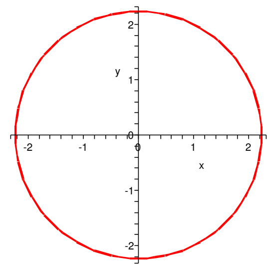
$$y - y_1 = m_1(x - x_1) \quad\Longrightarrow\quad y - 2 = \frac{1}{4}(x - 4) \quad\Longrightarrow\quad y = \frac{1}{4}x + 1 \quad \text{at} \quad (4, 2)$$

and

$$y - y_2 = m_2(x - x_2) \quad\Longrightarrow\quad y - (-2) = -\frac{1}{4}(x - 4) \quad\Longrightarrow\quad y = -\frac{1}{4}x - 1 \quad \text{at} \quad (4, -2)$$

EXAMPLE: If $x^2 + y^2 = 5$, find $\dfrac{dy}{dx}$. Then find an equation of the tangent line to $x^2 + y^2 = 5$ at the point $(2, 1)$.

Solution: The graph of the curve $x^2 + y^2 = 5$ is a circle:



To find $\dfrac{dy}{dx}$ we differentiate both sides:

$$x^2 + y^2 = 5 \implies (x^2 + y^2)' = 5' \implies (x^2)' + (y^2)' = 0 \implies 2x \cdot x' + 2y \cdot y' = 0$$

hence

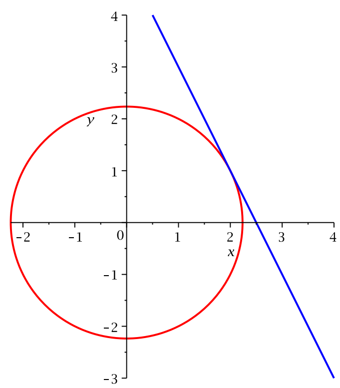
$$2x \cdot 1 + 2y \cdot y' = 0 \implies 2x + 2y \cdot y' = 0 \implies 2y \cdot y' = -2x \implies y' = -\frac{2x}{2y} = -\frac{x}{y}$$

To find an equation of the tangent line to $x^2 + y^2 = 5$ at the point $(2, 1)$ we note that

$$m = y'(2) = -\frac{2}{1} = -2$$

therefore

$$y - y_0 = m(x - x_0) \implies y - 1 = -2 \cdot (x - 2) \implies y = -2x + 5$$



EXAMPLE: If $2x^2 + 3y^2 = 5$, find $\dfrac{dy}{dx}$. Then find an equation of the tangent line to $2x^2 + 3y^2 = 5$ at the point $(1, 1)$.

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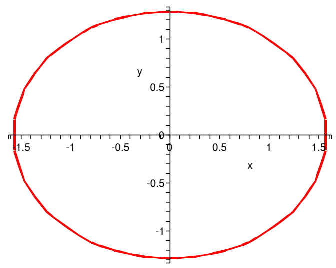
Solution: The graph of the curve $2x^2+3y^2=5$ is an ellipse:



To find $\dfrac{dy}{dx}$ we differentiate both sides:

$$2x^2+3y^2=5 \implies (2x^2+3y^2)'=5' \implies 2(x^2)'+3(y^2)'=0 \implies 2(2x\cdot x')+3(2y\cdot y')=0$$

hence

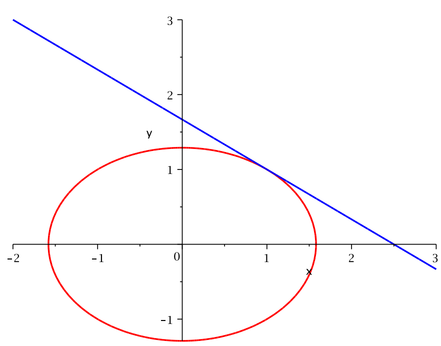
$$2(2x\cdot 1)+3(2y\cdot y')=0 \implies 4x+6y\cdot y'=0 \implies 6y\cdot y'=-4x \implies y'=-\frac{4x}{6y}=-\frac{2x}{3y}$$

To find an equation of the tangent line to $2x^2+3y^2=5$ at the point $(1,1)$ we note that

$$m=y'(1)=-\frac{2\cdot 1}{3\cdot 1}=-\frac{2}{3}$$

therefore

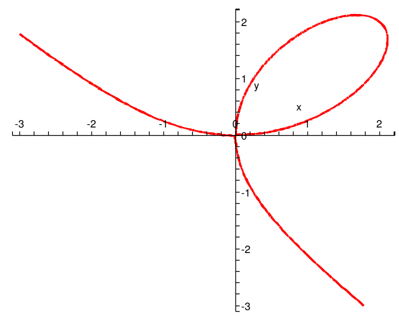
$$y-y_0=m(x-x_0) \implies y-1=-\frac{2}{3}(x-1) \implies y=-\frac{2}{3}x+\frac{5}{3}$$



EXAMPLE: If $x^3+y^3=4xy$, find $\dfrac{dy}{dx}$. Then find an equation of the tangent line to $x^3+y^3=4xy$ at the point $(2,2)$. At what point in the first quadrant is the tangent line horizontal?

EXAMPLE: If $x^3+y^3=4xy$, find $\dfrac{dy}{dx}$. Then find an equation of the tangent line to $x^3+y^3=4xy$ at the point $(2,2)$. At what point in the first quadrant is the tangent line horizontal?

Solution: The graph of the curve $x^3+y^3=4xy$ is the folium of Descartes:



(a) To find $\dfrac{dy}{dx}$ we differentiate both sides:

$$x^3+y^3=4xy \quad\Longrightarrow\quad (x^3+y^3)'=(4xy)' \quad\Longrightarrow\quad (x^3)'+(y^3)'=(4xy)'$$

Since $(4xy)'=4(xy)'=4(x'y+xy')=4(1\cdot y+xy')=4(y+xy')$, this gives us

$$3x^2\cdot x'+3y^2\cdot y'=4(y+xy') \quad\Longrightarrow\quad 3x^2+3y^2\cdot y'=4y+4xy' \quad\Longrightarrow\quad 3y^2y'-4xy'=4y-3x^2$$

hence

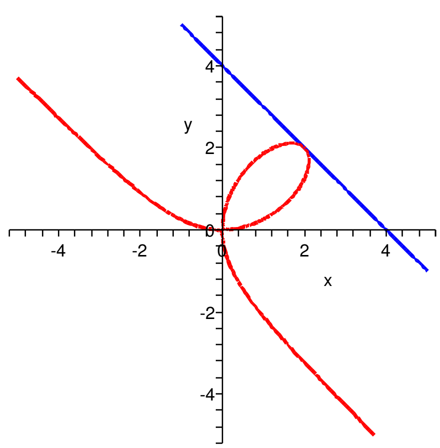
$$y'(3y^2-4x)=4y-3x^2 \quad\Longrightarrow\quad y'=\frac{4y-3x^2}{3y^2-4x}$$

(b) To find an equation of the tangent line to $x^3+y^3=4xy$ at the point $(2,2)$ we note that

$$m=y'(2)=\frac{4\cdot 2-3\cdot 2^2}{3\cdot 2^2-4\cdot 2}=-1$$

therefore

$$y-y_0=m(x-x_0) \quad\Longrightarrow\quad y-2=-1\cdot(x-2) \quad\Longrightarrow\quad y=-x+4$$



(c) The tangent line is horizontal if $y' = 0$. Using the expression for y' from part (a), we see that $y' = 0$ when $4y - 3x^2 = 0$ (provided that $3y^2 - 4x \neq 0$). We have

$$4y - 3x^2 = 0 \quad \Longrightarrow \quad y = \frac{3}{4}x^2 \quad \overset{x^3+y^3=4xy}{\Longrightarrow} \quad x^3 + \left(\frac{3}{4}x^2\right)^3 = 4x\left(\frac{3}{4}x^2\right) \quad \Longrightarrow \quad x^3 + \frac{27}{64}x^6 = 3x^3$$

which gives

$$\frac{27}{64}x^6 = 2x^3$$

Dividing both sides by 2, we get $\frac{27}{128}x^6 = x^3$. Since $x \neq 0$ in the first quadrant, we can divide both sides by x^3 which implies

$$\frac{27}{128}x^3 = 1 \quad \Longrightarrow \quad x^3 = \frac{128}{27}$$

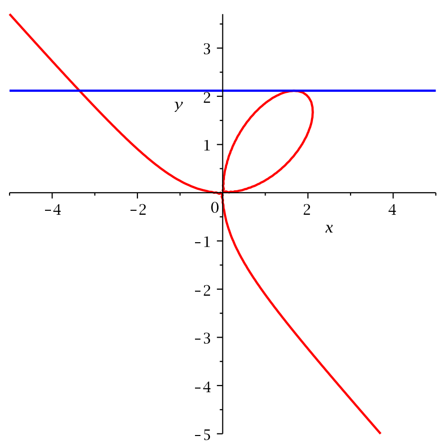
hence

$$x = \sqrt[3]{\frac{128}{27}} = \left\{\frac{\sqrt[3]{128}}{\sqrt[3]{27}} = \frac{\sqrt[3]{64 \cdot 2}}{\sqrt[3]{27}} = \frac{\sqrt[3]{64}\sqrt[3]{2}}{\sqrt[3]{27}} = \frac{4\sqrt[3]{2}}{3}\right\} = \frac{4}{3}\sqrt[3]{2} \approx 1.6798947$$

Plugging in this into $y = \frac{3}{4}x^2$, we get

$$y = \frac{3}{4}\left(\frac{4}{3}\sqrt[3]{2}\right)^2 = \frac{4}{3}\sqrt[3]{4} \approx 2.1165347$$

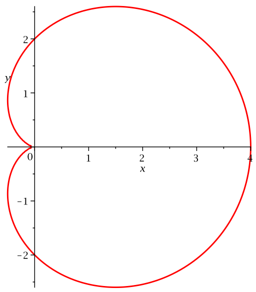
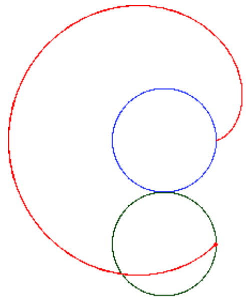
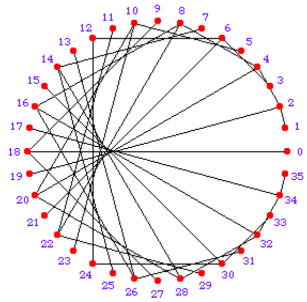
Finally, one can check that $3y^2 - 4x \neq 0$ at $\left(\frac{4}{3}\sqrt[3]{2}, \frac{4}{3}\sqrt[3]{4}\right)$. Thus the tangent line is horizontal at $\left(\frac{4}{3}\sqrt[3]{2}, \frac{4}{3}\sqrt[3]{4}\right)$, which is approximately $(1.6798947, 2.1165347)$. Looking at the figure, we see that our answer is reasonable.



EXAMPLE: If $(x^2+y^2-2x)^2 = 4(x^2+y^2)$, find an equation of the tangent line to $(x^2+y^2-2x)^2 = 4(x^2+y^2)$ at the point $(0, 2)$.

EXAMPLE: If $(x^2+y^2-2x)^2 = 4(x^2+y^2)$, find an equation of the tangent line to $(x^2+y^2-2x)^2 = 4(x^2+y^2)$ at the point $(0,2)$.

Solution: The graph of the curve $(x^2+y^2-2x)^2 = 4(x^2+y^2)$ is the cardioid:


We differentiate both sides of $(x^2+y^2-2x)^2 = 4(x^2+y^2)$:

$$[(x^2+y^2-2x)^2]' = [4(x^2+y^2)]' \quad \Longrightarrow \quad 2(x^2+y^2-2x)^{2-1}(x^2+y^2-2x)' = 4(x^2+y^2)'$$

hence

$$2(x^2+y^2-2x)(2x+2y\cdot y'-2) = 4(2x+2y\cdot y') \tag{3}$$

To find slope of the tangent line to $(x^2+y^2-2x)^2 = 4(x^2+y^2)$ at the point $(0,2)$ we replace x by 0 and y by 2 in (3):

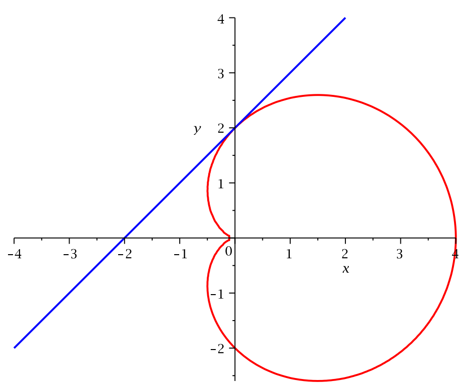
$$2(0^2+2^2-2\cdot 0)(2\cdot 0+2\cdot 2\cdot y'-2) = 4(2\cdot 0+2\cdot 2\cdot y') \quad \Longrightarrow \quad 8(4y'-2) = 4(4y')$$

hence

$$32y'-16 = 16y' \quad \Longrightarrow \quad 32y'-16y' = 16 \quad \Longrightarrow \quad 16y' = 16 \quad \Longrightarrow \quad y' = \frac{16}{16} = 1$$

so $m = 1$, therefore

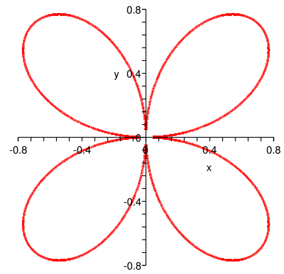
$$y - y_0 = m(x-x_0) \quad \Longrightarrow \quad y-2 = 1\cdot(x-0) \quad \Longrightarrow \quad y = x+2$$



REMARK: If we first find $\dfrac{dy}{dx}$ for any x and then find the slope of the tangent line to $(x^2+y^2-2x)^2 = 4(x^2+y^2)$ at the point $(0,2)$, the computations will be more complicated (see Appendix, page 10).

EXAMPLE: If $(x^2+y^2)^3=4x^2y^2$, find $\dfrac{dy}{dx}$.

Solution: The graph of the curve $(x^2+y^2)^3=4x^2y^2$ is a rose curve:



Solution: To find $\dfrac{dy}{dx}$ we differentiate both sides:

$$(x^2+y^2)^3=4x^2y^2 \quad\Longrightarrow\quad [(x^2+y^2)^3]'=(4x^2y^2)'$$

hence

$$3(x^2+y^2)^{3-1}\cdot(x^2+y^2)'=4(x^2y^2)' \tag{4}$$

Note that

$$(x^2+y^2)'=(x^2)'+(y^2)'=2x+2y\cdot y'=2(x+y\cdot y') \tag{5}$$

and

$$(x^2y^2)'=(x^2)'y^2+x^2(y^2)'=2xy^2+x^2\cdot 2y\cdot y'=2(xy^2+x^2y\cdot y') \tag{6}$$

Substituting (5) and (6) into (4), we obtain

$$3(x^2+y^2)^2\cdot 2(x+y\cdot y')=4\cdot 2(xy^2+x^2y\cdot y')$$

$$3(x^2+y^2)^2(x+y\cdot y')=4(xy^2+x^2y\cdot y')$$

$$3(x^4+2x^2y^2+y^4)(x+y\cdot y')=4xy^2+4x^2y\cdot y'$$

We now expand the parentheses and solve this equation for y':

$$3x^5+3x^4y\cdot y'+6x^3y^2+6x^2y^3\cdot y'+3xy^4+3y^5\cdot y'=4xy^2+4x^2y\cdot y'$$

so

$$3x^4y\cdot y'+6x^2y^3\cdot y'+3y^5\cdot y'-4x^2y\cdot y'=-3x^5-6x^3y^2-3xy^4+4xy^2$$

hence

$$y'(3x^4y+6x^2y^3+3y^5-4x^2y)=-3x^5-6x^3y^2-3xy^4+4xy^2$$

therefore

$$y'=\frac{-3x^5-6x^3y^2-3xy^4+4xy^2}{3x^4y+6x^2y^3+3y^5-4x^2y}=-\frac{3x^5+6x^3y^2+3y^4x-4y^2x}{3y^5+6y^3x^2+3x^4y-4x^2y}$$

EXAMPLE: Find y' if $\sin(x+y)=y^2\cos x$.

Solution: We have

$$\sin(x+y)=y^2\cos x \quad\Longrightarrow\quad (\sin(x+y))'=(y^2\cos x)'$$

therefore

$$\cos(x+y)\cdot(x+y)'=(y^2)'\cos x+y^2(\cos x)'$$

hence

$$\cos(x+y)\cdot(1+y')=2y\cdot y'\cos x-y^2\sin x$$

We now solve this equation for y':

$$\cos(x+y)+\cos(x+y)\cdot y'=2y\cdot y'\cos x-y^2\sin x$$

so

$$\cos(x+y)\cdot y'-2y\cdot y'\cos x=-\cos(x+y)-y^2\sin x$$

therefore

$$y'(\cos(x+y)-2y\cos x)=-\cos(x+y)-y^2\sin x$$

Dividing both sides by $\cos(x+y)-2y\cos x$, we get

$$y'=\frac{-\cos(x+y)-y^2\sin x}{\cos(x+y)-2y\cos x}=\frac{\cos(x+y)+y^2\sin x}{2y\cos x-\cos(x+y)}$$

Appendix

EXAMPLE: If $(x^2+y^2-2x)^2 = 4(x^2+y^2)$, find $\dfrac{dy}{dx}$.

Solution: To find $\dfrac{dy}{dx}$ we differentiate both sides:

$$(x^2+y^2-2x)^2 = 4(x^2+y^2) \quad \Longrightarrow \quad [(x^2+y^2-2x)^2]' = [4(x^2+y^2)]'$$

hence

$$2(x^2+y^2-2x)^{2-1} \cdot (x^2+y^2-2x)' = 4(x^2+y^2)' \tag{7}$$

Note that

$$(x^2+y^2-2x)' = (x^2)' + (y^2)' - (2x)' = 2x + 2y \cdot y' - 2 = 2(x + y \cdot y' - 1) \tag{8}$$

and

$$(x^2+y^2)' = (x^2)' + (y^2)' = 2x + 2y \cdot y' \tag{9}$$

Substituting (8) and (9) into (7), we obtain

$$2(x^2+y^2-2x) \cdot 2(x + y \cdot y' - 1) = 4(2x + 2y \cdot y')$$

$$(x^2+y^2-2x)(x + y \cdot y' - 1) = 2x + 2y \cdot y'$$

We now expand the parentheses and solve this equation for y':

$$x^3 + x^2y \cdot y' - x^2 + xy^2 + y^3 \cdot y' - y^2 - 2x^2 - 2xy \cdot y' + 2x = 2x + 2y \cdot y'$$

so

$$x^2y \cdot y' + y^3 \cdot y' - 2xy \cdot y' - 2y \cdot y' = -x^3 - xy^2 + 3x^2 + y^2$$

hence

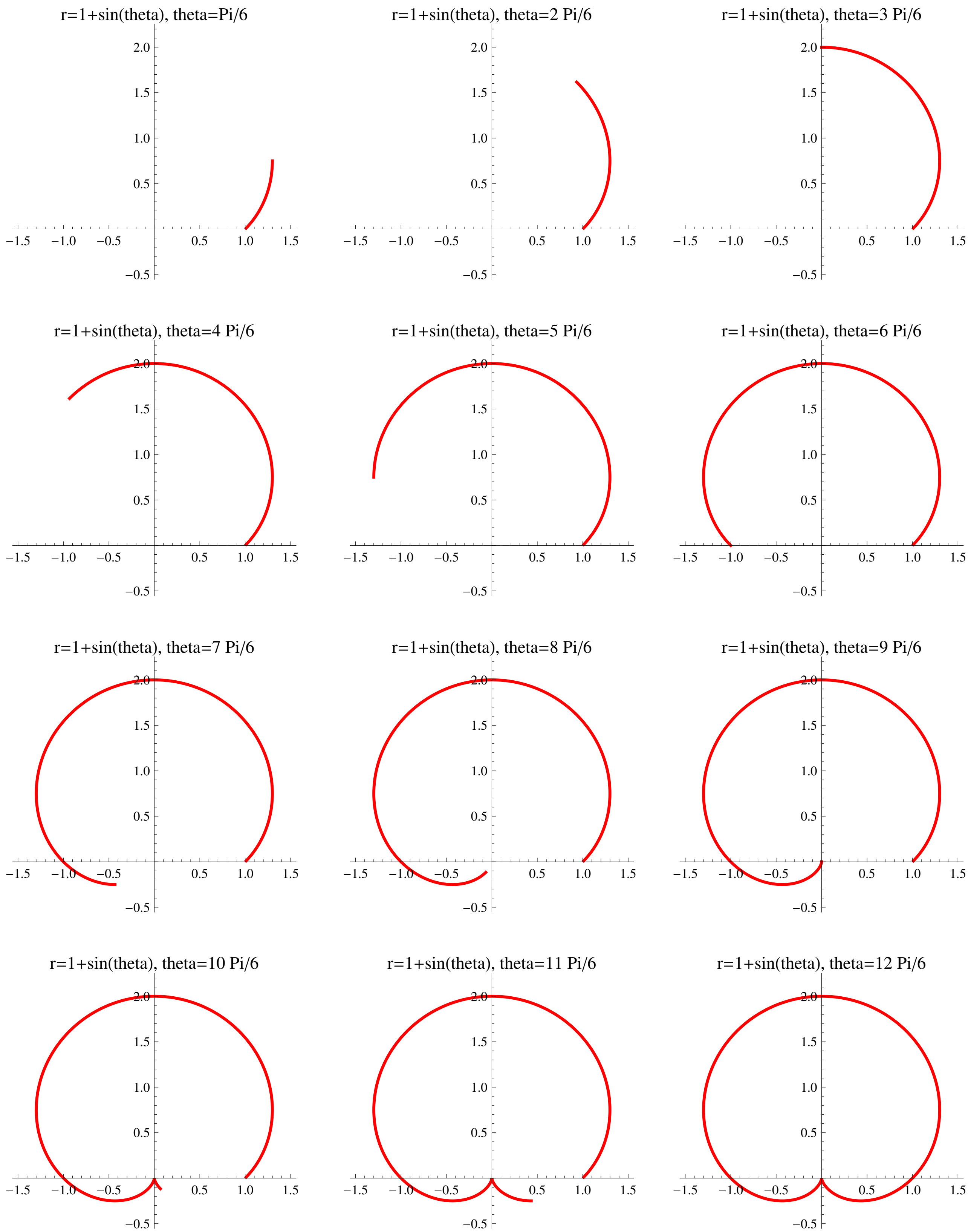
$$y'(x^2y + y^3 - 2xy - 2y) = -x^3 - xy^2 + 3x^2 + y^2$$

therefore

$$y' = \frac{-x^3 - xy^2 + 3x^2 + y^2}{x^2y + y^3 - 2xy - 2y} = -\frac{x^3 + xy^2 - 3x^2 - y^2}{y^3 + x^2y - 2xy - 2y}$$

EXAMPLE: Sketch the curve $r = 1 + \sin\theta,\ 0 \le \theta \le 2\pi$ (cardioid).

Solution: We have



EXAMPLE: Sketch the curve $r = 1 - \cos\theta,\ 0 \le \theta \le 2\pi$ (cardioid).

Solution: We have