The Derivative as a Function

DEFINITION: The **derivative of a function** f **at a number** a**,** denoted by $f'(a)$, is

$$\boxed{f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

if this limit exists.

REMARK: An equivalent way of stating the definition of the derivative is

$$\boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

OTHER NOTATIONS:

$$\boxed{f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}f(x)=Df(x)=D_xf(x)}$$

EXAMPLE: If $f(x)=3x-5$, find $f'(x)$.

Solution 1: We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\Big(3(a+h)-5\Big)-(3a-5)}{h}$$

$$=\lim_{h\to 0}\frac{3a+3h-5-3a+5}{h}=\lim_{h\to 0}\frac{3h}{h}=\lim_{h\to 0}3=3$$

Solution 2: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(3x-5)-(3a-5)}{x-a}$$

$$=\lim_{x\to a}\frac{3x-5-3a+5}{x-a}=\lim_{x\to a}\frac{3x-3a}{x-a}=\lim_{x\to a}\frac{3(x-a)}{x-a}=\lim_{x\to a}3=3$$

EXAMPLE: If $f(x)=x^2$, find $f'(x)$.

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Solution 1: We have

$$\begin{aligned} f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{(a+h)^2-a^2}{h} \\ &= \lim_{h\to 0}\frac{a^2+2ah+h^2-a^2}{h} \\ &= \lim_{h\to 0}\frac{2ah+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(2a+h)}{h} \\ &= \lim_{h\to 0}(2a+h) \\ &= 2a+0 \\ &= 2a \end{aligned}$$

or

$$\begin{aligned} f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{(a+h)^2-a^2}{h} \\ &= \lim_{h\to 0}\frac{(a+h-a)(a+h+a)}{h} \\ &= \lim_{h\to 0}\frac{h(2a+h)}{h} \\ &= \lim_{h\to 0}(2a+h) \\ &= 2a+0 \\ &= 2a \end{aligned}$$

Solution 2: We have

$$\begin{aligned} f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{x^2-a^2}{x-a} \\ &= \lim_{x\to a}\frac{(x-a)(x+a)}{x-a} \\ &= \lim_{x\to a}(x+a) \\ &= a+a \\ &= 2a \end{aligned}$$

EXAMPLE: If $f(x)=\sqrt{x}$, find $f'(x)$.

Solution 1: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}{(x-a)(\sqrt{x}+\sqrt{a})}$$

$$=\lim_{x\to a}\frac{(\sqrt{x})^2-(\sqrt{a})^2}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}$$

$$=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

Solution 1′: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a}=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x})^2-(\sqrt{a})^2}$$

$$=\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}=\lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

Solution 2: We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}{h(\sqrt{a+h}+\sqrt{a})}$$

$$=\lim_{h\to 0}\frac{(\sqrt{a+h})^2-(\sqrt{a})^2}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})}=\lim_{h\to 0}\frac{h}{h(\sqrt{a+h}+\sqrt{a})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{\sqrt{a+0}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

Solution 2′: We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{a+h-a}$$

$$=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{(\sqrt{a+h})^2-(\sqrt{a})^2}=\lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{(\sqrt{a+h}-\sqrt{a})(\sqrt{a+h}+\sqrt{a})}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{a+h}+\sqrt{a}}=\frac{1}{\sqrt{a+0}+\sqrt{a}}=\frac{1}{\sqrt{a}+\sqrt{a}}=\frac{1}{2\sqrt{a}}$$

EXAMPLE: If $f(x)=2x^3-5x$, find $f'(x)$.

Solution: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{(2x^3-5x)-(2a^3-5a)}{x-a}=\lim_{x\to a}\frac{2x^3-5x-2a^3+5a}{x-a}$$

$$=\lim_{x\to a}\frac{2x^3-2a^3-5x+5a}{x-a}=\lim_{x\to a}\frac{2(x^3-a^3)-5(x-a)}{x-a}$$

$$=\lim_{x\to a}\frac{2(x-a)(x^2+xa+a^2)-5(x-a)}{x-a}=\lim_{x\to a}\frac{(x-a)\Big(2(x^2+xa+a^2)-5\Big)}{x-a}$$

$$=\lim_{x\to a}\Big(2(x^2+xa+a^2)-5\Big)=2(a^2+a\cdot a+a^2)-5=2(3a^2)-5=6a^2-5$$

EXAMPLE: If $f(x)=\dfrac{x+3}{4-x}$, find $f'(x)$.

Solution 1: We have

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\dfrac{x+3}{4-x}-\dfrac{a+3}{4-a}}{x-a}=\lim_{x\to a}\frac{\left(\dfrac{x+3}{4-x}-\dfrac{a+3}{4-a}\right)\cdot(4-x)(4-a)}{(x-a)\cdot(4-x)(4-a)}$$

$$=\lim_{x\to a}\frac{\dfrac{x+3}{4-x}\cdot(4-x)(4-a)-\dfrac{a+3}{4-a}\cdot(4-x)(4-a)}{(x-a)\cdot(4-x)(4-a)}$$

$$=\lim_{x\to a}\frac{(x+3)(4-a)-(a+3)(4-x)}{(x-a)(4-x)(4-a)}=\lim_{x\to a}\frac{(4x-xa+12-3a)-(4a-ax+12-3x)}{(x-a)(4-x)(4-a)}$$

$$=\lim_{x\to a}\frac{4x-xa+12-3a-4a+ax-12+3x}{(x-a)(4-x)(4-a)}=\lim_{x\to a}\frac{7x-7a}{(x-a)(4-x)(4-a)}$$

$$=\lim_{x\to a}\frac{7(x-a)}{(x-a)(4-x)(4-a)}=\lim_{x\to a}\frac{7}{(4-x)(4-a)}=\frac{7}{(4-a)(4-a)}=\frac{7}{(4-a)^2}$$

Solution 2: We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{h\to 0}\frac{\dfrac{(a+h)+3}{4-(a+h)}-\dfrac{a+3}{4-a}}{h}=\lim_{h\to 0}\frac{\dfrac{a+h+3}{4-a-h}-\dfrac{a+3}{4-a}}{h}$$

$$=\lim_{h\to 0}\frac{\left(\dfrac{a+h+3}{4-a-h}-\dfrac{a+3}{4-a}\right)\cdot(4-a-h)(4-a)}{h\cdot(4-a-h)(4-a)}$$

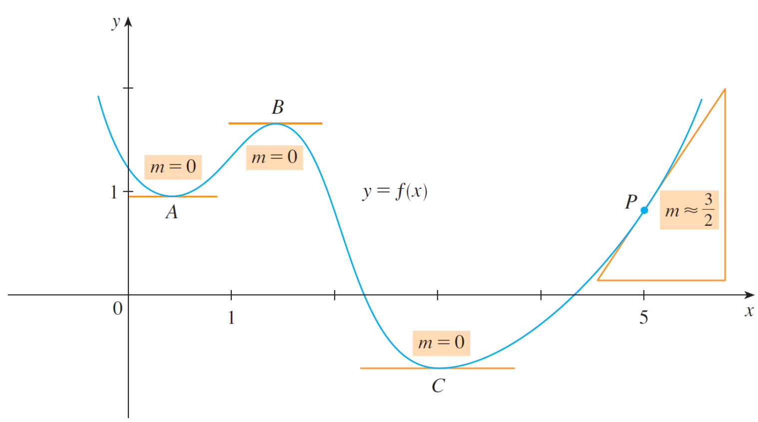
$$=\lim_{h\to 0}\frac{\dfrac{a+h+3}{4-a-h}\cdot(4-a-h)(4-a)-\dfrac{a+3}{4-a}\cdot(4-a-h)(4-a)}{h\cdot(4-a-h)(4-a)}$$

$$=\lim_{h\to 0}\frac{(a+h+3)(4-a)-(a+3)(4-a-h)}{h(4-a-h)(4-a)}$$

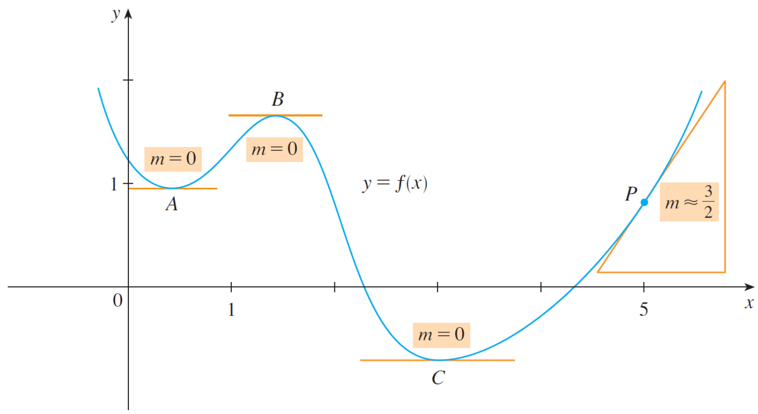
$$=\lim_{h\to 0}\frac{(4a-a^2+4h-ah+12-3a)-(4a-a^2-ah+12-3a-3h)}{h(4-a-h)(4-a)}$$

$$=\lim_{h\to 0}\frac{7h}{h(4-a-h)(4-a)}=\lim_{h\to 0}\frac{7}{(4-a-h)(4-a)}=\frac{7}{(4-a-0)(4-a)}=\frac{7}{(4-a)^2}$$

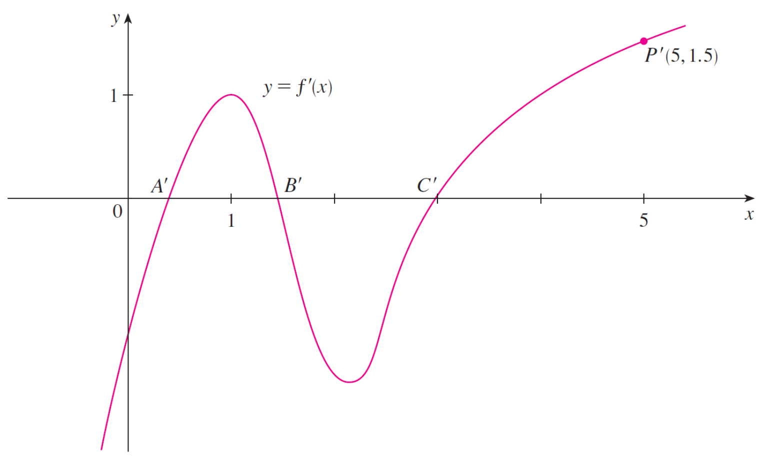
EXAMPLE: Use the graph of a function f to sketch the graph of the derivative f'.



EXAMPLE: Use the graph of a function f to sketch the graph of the derivative f'.



Solution: We have



DEFINITION: A function f is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLES:

1. A polynomial, $\sin x$, $\cos x$ are differentiable everywhere.

2. The function $f(x) = \dfrac{1}{x}$ in differentiable on $(-\infty, 0)$ and on $(0, \infty)$.

EXAMPLE: Show that $f(x) = |x|$ is not differentiable at $x = 0$.

EXAMPLE: Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Solution: Note that

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

Therefore on the one hand we have

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

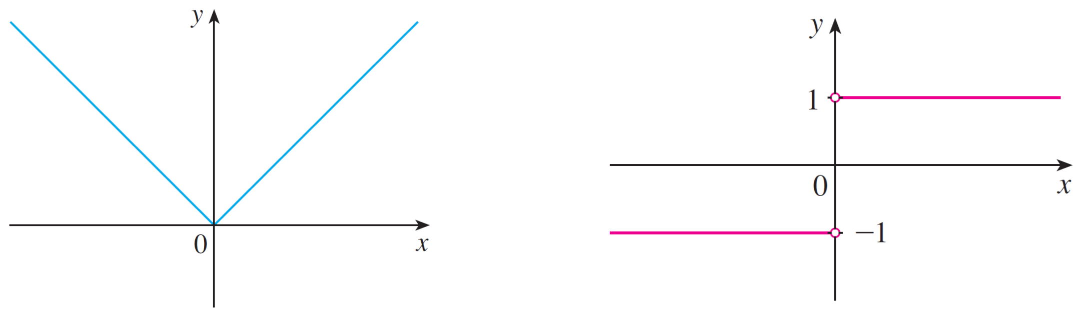
On the other hand,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

Since

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \ne \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h}$$

it follows that $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist. Therefore $f'(0)$ does not exist, so $f(x) = |x|$ is not differentiable at $x = 0$.



(a) $y = f(x) = |x|$ (b) $y = f'(x)$

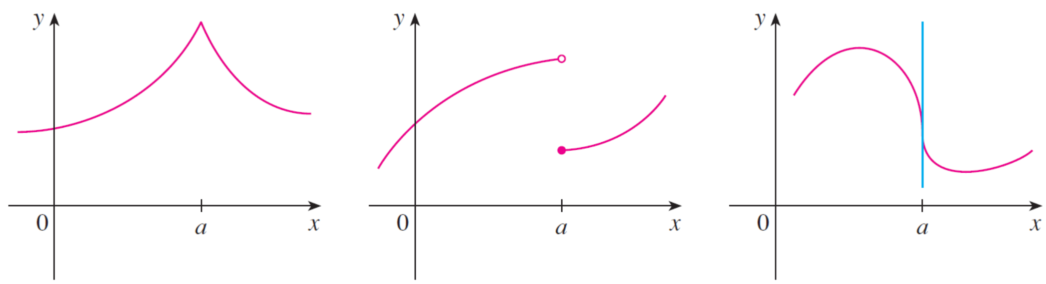
How can a function fail to be differentiable?

There are three main instances when f is not differentiable at a:

1. When the graph of f has a "corner" or "kink" in it.

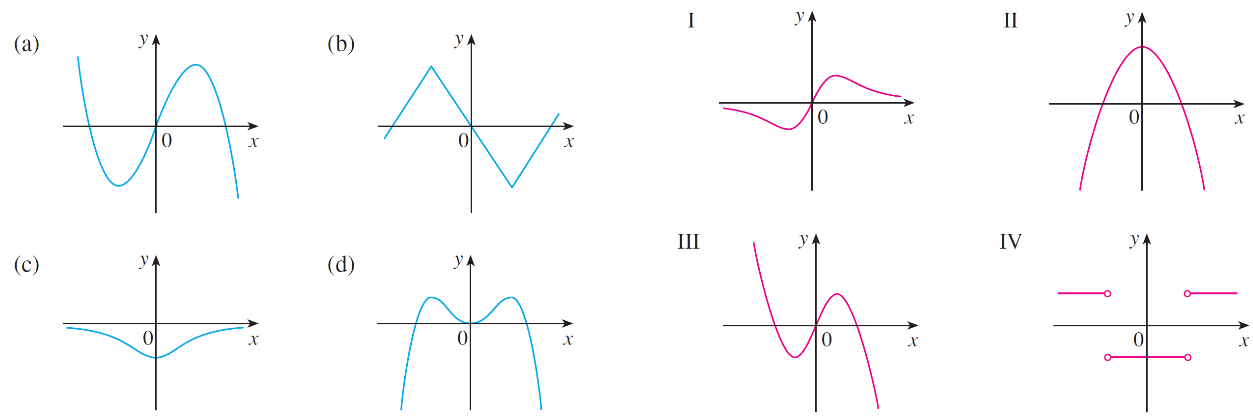
2. When f is not continuous at a.

3. When the graph of f has a **vertical tangent line** at $x = a$; that is, when f is continuous at a and $\lim_{x\to a} |f'(x)| = \infty$.



(a) A corner (b) A discontinuity (c) A vertical tangent

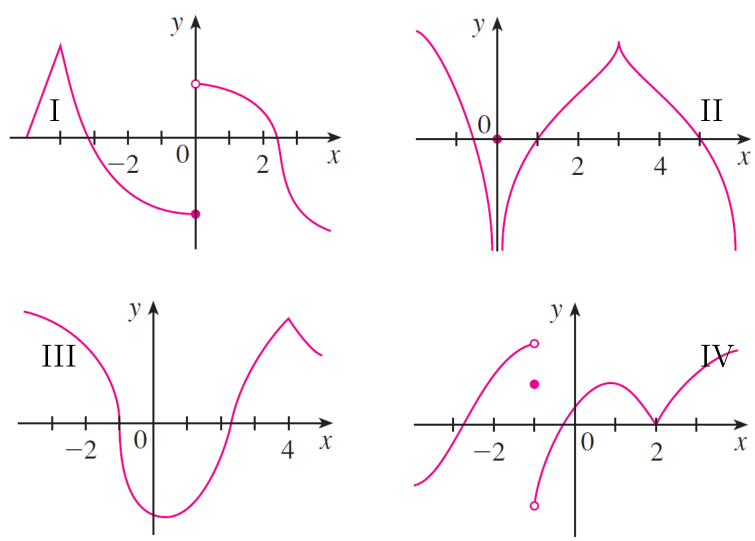
EXAMPLE: Match the graph of each function in (a)-(d) with the graph of its derivative in I-IV. Give reasons for your choice.



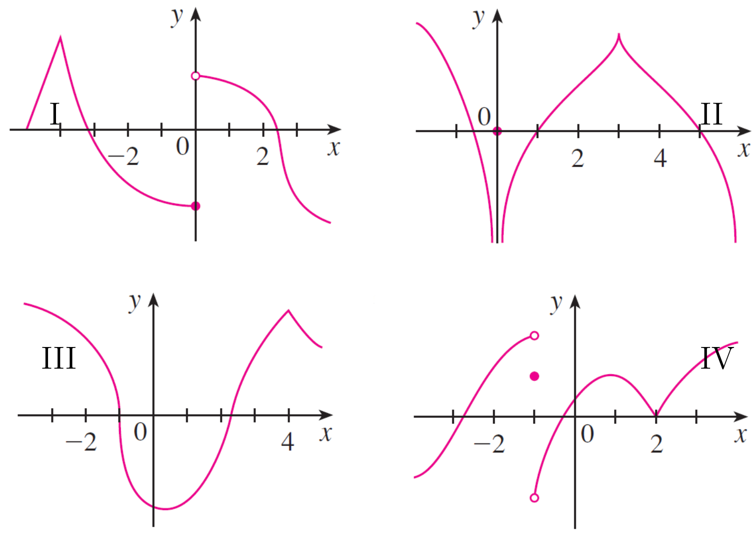
THEOREM: If f is differentiable at a, then f is continuous at a.

REMARK: The opposite statement is not true! That is, there are functions that are continuous at a certain point but not differentiable. For example, $f(x) = |x|$.

EXAMPLE: The graph is given. State, with reasons, the numbers at which f is not differentiable.



EXAMPLE: The graph is given. State, with reasons, the numbers at which f is not differentiable.



Solution:

(I) f is not differentiable at $x=-4$ (corner) and $x=0$ (discontinuity)

(II) f is not differentiable at $x=0$ (discontinuity) and $x=3$ (corner)

(III) f is not differentiable at $x=-1$ (vertical tangent) and $x=4$ (corner)

(IV) f is not differentiable at $x=-1$ (discontinuity) and $x=2$ (corner)

Higher derivatives

If f is a differentiable function, then f' is also a function. So, f' may have a derivative of its own, denoted by $(f')'=f''$. This new function f'' is called the **second derivative** of f.

OTHER NOTATIONS: $\boxed{f''(x)=y''=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}}$

EXAMPLE: We already know that if $f(x)=2x^3-5x$, then $f'(x)=6x^2-5$. The second derivative is

$$\begin{aligned} f''(a) &= \lim_{h\to 0}\frac{f'(a+h)-f'(a)}{h}=\lim_{h\to 0}\frac{6(a+h)^2-5-(6a^2-5)}{h} \\ &= \lim_{h\to 0}\frac{6a^2+12ah+6h^2-5-6a^2+5}{h}=\lim_{h\to 0}\frac{12ah+6h^2}{h} \\ &= \lim_{h\to 0}\frac{h(12a+6h)}{h}=\lim_{h\to 0}(12a+6h)=12a \end{aligned}$$

The **third derivative** f''' is the derivative of the second derivative $f''' = (f'')'$.

OTHER NOTATIONS:

$$\boxed{f'''(x) = y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}}$$

Similarly, the **fourth derivative** f'''' is the derivative of the third derivative $f'''' = (f''')'$. And so on.

REMARK: The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general,

$$\boxed{f^{(n)}(x) = y^{(n)} = \frac{d^ny}{dx^n}}$$

Application

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$\boxed{a(t) = v'(t) = s''(t)}$$

or, in Leibniz notation,

$$\boxed{a = \frac{dv}{dt} = \frac{d^2s}{dt^2}}$$

Appendix

EXAMPLE: If $f(x) = 3 - 2x - 7x^2$, find $f'(x)$.

Solution 1: We have

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\Big(3 - 2(a+h) - 7(a+h)^2\Big) - \Big(3 - 2a - 7a^2\Big)}{h}$$

$$= \lim_{h\to 0} \frac{3 - 2a - 2h - 7(a^2 + 2ah + h^2) - 3 + 2a + 7a^2}{h}$$

$$= \lim_{h\to 0} \frac{3 - 2a - 2h - 7a^2 - 14ah - 7h^2 - 3 + 2a + 7a^2}{h}$$

$$= \lim_{h\to 0} \frac{-2h - 14ah - 7h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-2 - 14a - 7h)}{h}$$

$$= \lim_{h\to 0} (-2 - 14a - 7h)$$

$$= -2 - 14a - 7 \cdot 0$$

$$= -2 - 14a$$

Solution 2: We have

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{(3 - 2x - 7x^2) - (3 - 2a - 7a^2)}{x - a}$$

$$= \lim_{x\to a} \frac{3 - 2x - 7x^2 - 3 + 2a + 7a^2}{x - a}$$

$$= \lim_{x\to a} \frac{-2x - 7x^2 + 2a + 7a^2}{x - a}$$

$$= \lim_{x\to a} \frac{-2x + 2a - 7x^2 + 7a^2}{x - a}$$

$$= \lim_{x\to a} \frac{-2(x - a) - 7(x^2 - a^2)}{x - a}$$

$$= \lim_{x\to a} \frac{-2(x - a) - 7(x - a)(x + a)}{x - a}$$

$$= \lim_{x\to a} \frac{(x - a)\Big(-2 - 7(x + a)\Big)}{x - a}$$

$$= \lim_{x\to a} \Big(-2 - 7(x + a)\Big)$$

$$= -2 - 7(a + a) = -2 - 7(2a) = -2 - 14a$$

EXAMPLE: If $f(x) = \sqrt{5x+9}$, find $f'(x)$.

Solution 1: We have

$$\begin{aligned}
f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{\sqrt{5x+9}-\sqrt{5a+9}}{x-a} \\
&= \lim_{x\to a}\frac{(\sqrt{5x+9}-\sqrt{5a+9})(\sqrt{5x+9}+\sqrt{5a+9})}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} \\
&= \lim_{x\to a}\frac{(\sqrt{5x+9})^2-(\sqrt{5a+9})^2}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} \\
&= \lim_{x\to a}\frac{(5x+9)-(5a+9)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} \\
&= \lim_{x\to a}\frac{5x+9-5a-9}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} \\
&= \lim_{x\to a}\frac{5x-5a}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} \\
&= \lim_{x\to a}\frac{5(x-a)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} \\
&= \lim_{x\to a}\frac{5}{\sqrt{5x+9}+\sqrt{5a+9}} \\
&= \frac{5}{\sqrt{5a+9}+\sqrt{5a+9}} \\
&= \frac{5}{2\sqrt{5a+9}}
\end{aligned}$$

In short,

$$f'(a) = \lim_{x\to a}\frac{\sqrt{5x+9}-\sqrt{5a+9}}{x-a} = \lim_{x\to a}\frac{(\sqrt{5x+9}-\sqrt{5a+9})(\sqrt{5x+9}+\sqrt{5a+9})}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$= \lim_{x\to a}\frac{(\sqrt{5x+9})^2-(\sqrt{5a+9})^2}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} = \lim_{x\to a}\frac{(5x+9)-(5a+9)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})}$$

$$= \lim_{x\to a}\frac{5(x-a)}{(x-a)(\sqrt{5x+9}+\sqrt{5a+9})} = \lim_{x\to a}\frac{5}{\sqrt{5x+9}+\sqrt{5a+9}} = \frac{5}{2\sqrt{5a+9}}$$

Solution 2: We have

$$
\begin{aligned}
f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{\sqrt{5(a+h)+9}-\sqrt{5a+9}}{h}\\
&= \lim_{h\to 0}\frac{(\sqrt{5(a+h)+9}-\sqrt{5a+9})(\sqrt{5(a+h)+9}+\sqrt{5a+9})}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})}\\
&= \lim_{h\to 0}\frac{(\sqrt{5(a+h)+9})^2-(\sqrt{5a+9})^2}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})}\\
&= \lim_{h\to 0}\frac{(5(a+h)+9)-(5a+9)}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})}\\
&= \lim_{h\to 0}\frac{5a+5h+9-5a-9}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})}\\
&= \lim_{h\to 0}\frac{5h}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})}\\
&= \lim_{h\to 0}\frac{5}{\sqrt{5(a+h)+9}+\sqrt{5a+9}}\\
&= \frac{5}{\sqrt{5(a+0)+9}+\sqrt{5a+9}}\\
&= \frac{5}{\sqrt{5a+9}+\sqrt{5a+9}}\\
&= \frac{5}{2\sqrt{5a+9}}
\end{aligned}
$$

In short,

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0}\frac{\sqrt{5(a+h)+9}-\sqrt{5a+9}}{h} = \lim_{h\to 0}\frac{(\sqrt{5(a+h)+9}-\sqrt{5a+9})(\sqrt{5(a+h)+9}+\sqrt{5a+9})}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})}\\
&= \lim_{h\to 0}\frac{(\sqrt{5(a+h)+9})^2-(\sqrt{5a+9})^2}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} = \lim_{h\to 0}\frac{(5(a+h)+9)-(5a+9)}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})}\\
&= \lim_{h\to 0}\frac{5h}{h(\sqrt{5(a+h)+9}+\sqrt{5a+9})} = \lim_{h\to 0}\frac{5}{\sqrt{5(a+h)+9}+\sqrt{5a+9}} = \frac{5}{2\sqrt{5a+9}}
\end{aligned}
$$

EXAMPLE: If $f(x)=\sqrt{2-3x}$, find $f'(x)$.

Solution: We have

$$
\begin{aligned}
f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}&=\lim_{x\to a}\frac{\sqrt{2-3x}-\sqrt{2-3a}}{x-a}\\
&=\lim_{x\to a}\frac{(\sqrt{2-3x}-\sqrt{2-3a})(\sqrt{2-3x}+\sqrt{2-3a})}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{(\sqrt{2-3x})^2-(\sqrt{2-3a})^2}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{(2-3x)-(2-3a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{2-3x-2+3a}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{-3x+3a}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{-3(x-a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{-3}{\sqrt{2-3x}+\sqrt{2-3a}}\\
&=\frac{-3}{\sqrt{2-3a}+\sqrt{2-3a}}\\
&=-\frac{3}{2\sqrt{2-3a}}
\end{aligned}
$$

In short,

$$
\begin{aligned}
f'(a)&=\lim_{x\to a}\frac{\sqrt{2-3x}-\sqrt{2-3a}}{x-a}=\lim_{x\to a}\frac{(\sqrt{2-3x}-\sqrt{2-3a})(\sqrt{2-3x}+\sqrt{2-3a})}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{(\sqrt{2-3x})^2-(\sqrt{2-3a})^2}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}=\lim_{x\to a}\frac{(2-3x)-(2-3a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}\\
&=\lim_{x\to a}\frac{-3(x-a)}{(x-a)(\sqrt{2-3x}+\sqrt{2-3a})}=\lim_{x\to a}\frac{-3}{\sqrt{2-3x}+\sqrt{2-3a}}=-\frac{3}{2\sqrt{2-3a}}
\end{aligned}
$$