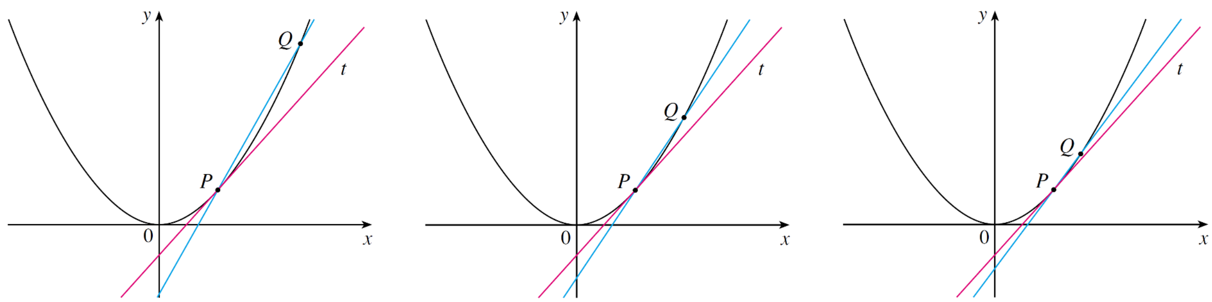
Derivatives and Rates of Change

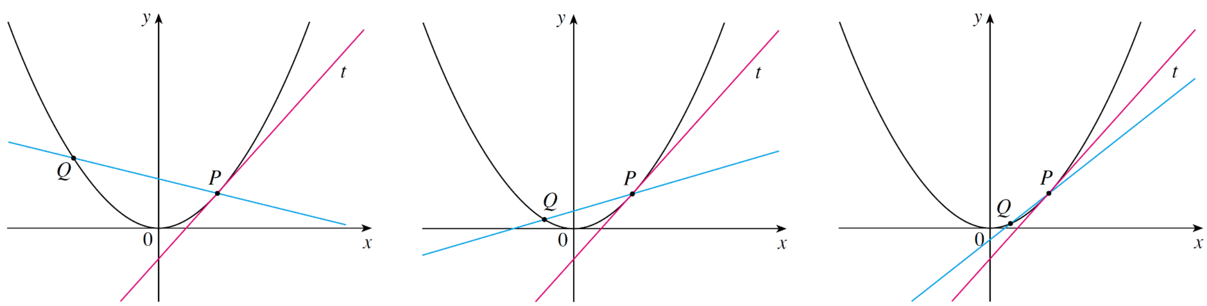
The Tangent Problem

EXAMPLE: Graph the parabola $y = x^2$ and the tangent line at the point $P(1, 1)$.

Solution: We have:



Q approaches P from the right



Q approaches P from the left

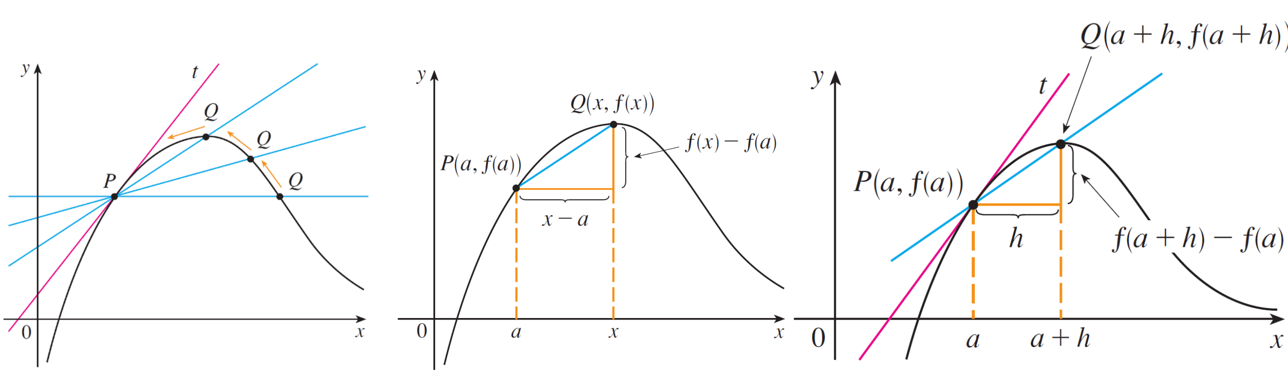
DEFINITION: The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$\boxed{m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} \tag{1}$$

provided that this limit exists.

There is another (equivalent) expression for the slope of the tangent line:

$$\boxed{m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}} \tag{2}$$



EXAMPLE: Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

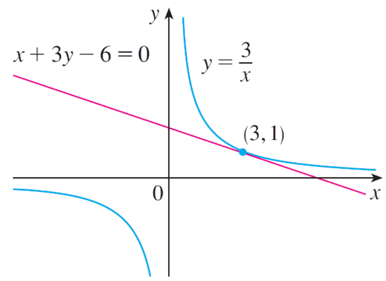
Solution 1: Let $f(x) = 3/x$. Then the slope of the tangent line at $(3, 1)$ is

$$m = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \overset{f(x)=3/x}{=} \lim_{x\to a}\frac{\dfrac{3}{x}-\dfrac{3}{a}}{x-a} = \lim_{x\to a}\frac{xa\cdot\left(\dfrac{3}{x}-\dfrac{3}{a}\right)}{xa\cdot(x-a)} = \lim_{x\to a}\frac{xa\cdot\dfrac{3}{x}-xa\cdot\dfrac{3}{a}}{xa(x-a)}$$

$$= \lim_{x\to a}\frac{3a-3x}{xa(x-a)} = \lim_{x\to a}\frac{3(a-x)}{xa(x-a)} = \lim_{x\to a}\frac{-3(x-a)}{xa(x-a)} = \lim_{x\to a}\frac{-3}{xa} = \frac{-3}{a^2} \overset{a=3}{=} \frac{-3}{3^2} = -\frac{1}{3}$$

Recall, that the point-slope equation of a line is

$$\boxed{y - y_0 = m(x - x_0)}$$



Therefore, an equation of the tangent line at the point $(3, 1)$ is

$$y - 1 = -\frac{1}{3}(x-3)$$

which simplifies to

$$x + 3y - 6 = 0$$

Solution 2: Equivalently, if we use formula (2), we get

$$m = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \overset{f(x)=3/x}{=} \lim_{h\to 0}\frac{\dfrac{3}{a+h}-\dfrac{3}{a}}{h} = \lim_{h\to 0}\frac{(a+h)a\cdot\left(\dfrac{3}{a+h}-\dfrac{3}{a}\right)}{(a+h)a\cdot h}$$

$$= \lim_{h\to 0}\frac{(a+h)a\cdot\dfrac{3}{a+h}-(a+h)a\cdot\dfrac{3}{a}}{(a+h)ah} = \lim_{h\to 0}\frac{3a-3(a+h)}{(a+h)ah} = \lim_{h\to 0}\frac{3a-3a-3h}{(a+h)ah}$$

$$= \lim_{h\to 0}\frac{-3h}{(a+h)ah} = \lim_{h\to 0}\frac{-3}{(a+h)a} = \frac{-3}{(a+0)a} = \frac{-3}{a^2} \overset{a=3}{=} \frac{-3}{3^2} = -\frac{1}{3}$$

and the same result follows.

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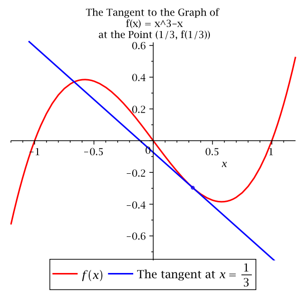
Solution 1: Let $f(x) = x^3 - x$. Then the slope of the tangent line at $(1/3, f(1/3))$ is

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \overset{f(x)=x^3-x}{=} \lim_{x\to a} \frac{(x^3 - x) - (a^3 - a)}{x - a} = \lim_{x\to a} \frac{x^3 - x - a^3 + a}{x - a}$$

$$= \lim_{x\to a} \frac{x^3 - a^3 - x + a}{x - a} = \lim_{x\to a} \frac{(x^3 - a^3) - (x - a)}{x - a} = \lim_{x\to a} \frac{(x - a)(x^2 + xa + a^2) - (x - a)\cdot 1}{x - a}$$

$$= \lim_{x\to a} \frac{(x - a)(x^2 + xa + a^2 - 1)}{x - a} = \lim_{x\to a}(x^2 + xa + a^2 - 1)$$

$$= a^2 + a^2 + a^2 - 1 = 3a^2 - 1 \overset{a=1/3}{=} 3\cdot\left(\frac{1}{3}\right)^2 - 1 = -\frac{2}{3}$$



Recall, that the point-slope equation of a line is

$$y - y_0 = m(x - x_0)$$

Therefore, an equation of the tangent line at the point $(1/3, f(1/3))$ is

$$y - y_0 = m(x - x_0) \quad\Longrightarrow\quad y - f(1/3) = -\frac{2}{3}\left(x - \frac{1}{3}\right) \quad\Longrightarrow\quad y + \frac{8}{27} = -\frac{2}{3}\left(x - \frac{1}{3}\right)$$

which simplifies to

$$18x + 27y + 2 = 0$$

Solution 2: Equivalently, if we use formula (2), we get

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} \overset{f(x)=x^3-x}{=} \lim_{h\to 0} \frac{[(a + h)^3 - (a + h)] - [a^3 - a]}{h}$$

$$= \lim_{h\to 0} \frac{[a^3 + 3a^2h + 3ah^2 + h^3 - a - h] - [a^3 - a]}{h} = \lim_{h\to 0} \frac{3a^2h + 3ah^2 + h^3 - h}{h}$$

$$= \lim_{h\to 0} \frac{h(3a^2 + 3ah + h^2 - 1)}{h} = \lim_{h\to 0}(3a^2 + 3ah + h^2 - 1) = 3a^2 - 1 \overset{a=1/3}{=} 3\cdot\left(\frac{1}{3}\right)^2 - 1 = -\frac{2}{3}$$

and the same result follows.

Solution 3: Similarly, if we use formula (2), we get

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h} \overset{f(x)=x^3-x}{=} \lim_{h\to 0} \frac{[(a + h)^3 - (a + h)] - [a^3 - a]}{h}$$

$$= \lim_{h\to 0} \frac{[(a + h)^3 - a^3] - [(a + h) - a]}{h} = \lim_{h\to 0} \frac{[((a + h) - a)((a + h)^2 + (a + h)a + a^2)] - [(a + h) - a]}{h}$$

$$= \lim_{h\to 0} \frac{h((a + h)^2 + (a + h)a + a^2) - h}{h} = \lim_{h\to 0} \frac{h[(a + h)^2 + (a + h)a + a^2 - 1]}{h}$$

$$= \lim_{h\to 0}((a + h)^2 + (a + h)a + a^2 - 1) = a^2 + a^2 + a^2 - 1 = 3a^2 - 1 \overset{a=1/3}{=} 3\cdot\left(\frac{1}{3}\right)^2 - 1 = -\frac{2}{3}$$

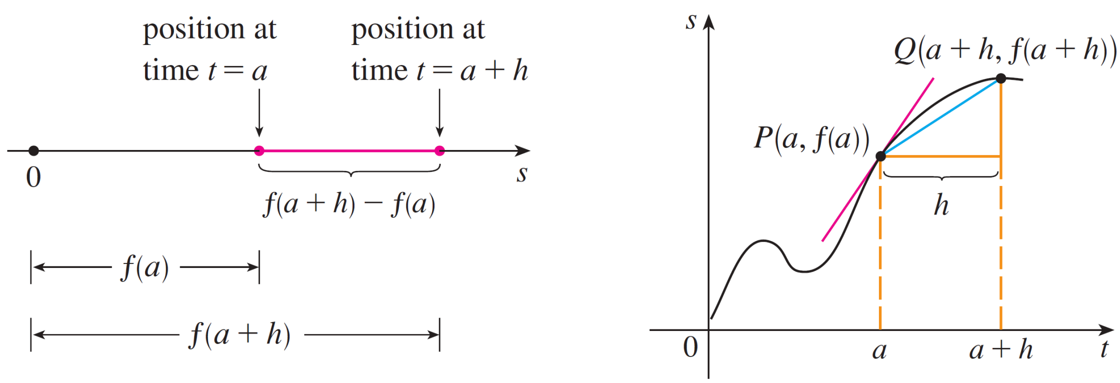
and the same result follows.

The Velocity Problem

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a+h) - f(a)$. The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in the second figure.



Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a+h]$. In other words, we let h approach 0. We define **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

$$\boxed{v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

REMARK: Equivalently,

$$\boxed{v(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

EXAMPLE: Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

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Solution:

(a) We use the equation of motion

$$\boxed{s = f(t) = 4.9t^2}$$

where t is time (in seconds) and s is the displacement (in meters) to find the velocity $v(a)$ after a seconds:

$$\begin{aligned} v(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{4.9x^2-4.9a^2}{x-a} \\ &= \lim_{x\to a}\frac{4.9(x^2-a^2)}{x-a} \\ &= \lim_{x\to a}\frac{4.9(x-a)(x+a)}{x-a} \\ &= \lim_{x\to a}4.9(x+a) \\ &= 4.9(a+a) \\ &= 4.9(2a) \\ &= 9.8a \end{aligned}$$

or

$$\begin{aligned} v(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0}\frac{4.9(a+h)^2-4.9a^2}{h} \\ &= \lim_{h\to 0}\frac{4.9(a^2+2ah+h^2)-4.9a^2}{h} \\ &= \lim_{h\to 0}\frac{4.9(a^2+2ah+h^2-a^2)}{h} \\ &= \lim_{h\to 0}\frac{4.9(2ah+h^2)}{h} \\ &= \lim_{h\to 0}\frac{4.9h(2a+h)}{h} \\ &= \lim_{h\to 0}4.9(2a+h) \\ &= 4.9(2a+0) \\ &= 4.9(2a) \\ &= 9.8a \end{aligned}$$

Therefore the velocity after 5 seconds is $v(5) = 9.8\cdot 5 = 49$ m/s.

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t_1 when $s(t_1) = 450$, that is,

$$4.9t_1^2 = 450 \quad\Longrightarrow\quad t_1^2 = \frac{450}{4.9} \quad\Longrightarrow\quad t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

The velocity of the ball as it hits the ground is therefore

$$v(t_1) = 9.8t_1 = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$

Derivatives

We have seen that limits of the form

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad \text{or} \quad \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

arise in finding the slope of a tangent line or the velocity of an object. Moreover, the same type of limit arises whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

DEFINITION: The **derivative of a function $\boldsymbol{f}$ at a number $\boldsymbol{a}$**, denoted by $f'(a)$, is

$$\boxed{f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

if this limit exists.

REMARK: Equivalently,

$$\boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

EXAMPLE: Find the derivative of the function $y=x^2-8x+9$ at the number a. Then find an equation of the tangent line to the parabola $y=x^2-8x+9$ at the point $(3,-6)$.

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Solution: We have

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{(x^2-8x+9)-(a^2-8a+9)}{x-a} \\ &= \lim_{x\to a}\frac{x^2-8x+9-a^2+8a-9}{x-a} \\ &= \lim_{x\to a}\frac{x^2-8x-a^2+8a}{x-a} \\ &= \lim_{x\to a}\frac{x^2-a^2-8x+8a}{x-a} \\ &= \lim_{x\to a}\frac{(x-a)(x+a)-8(x-a)}{x-a} \\ &= \lim_{x\to a}\frac{(x-a)[(x+a)-8]}{x-a} \\ &= \lim_{x\to a}[(x+a)-8] \\ &= 2a-8 \end{aligned}$$

or

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\ &= \lim_{h\to 0}\frac{[(a+h)^2-8(a+h)+9]-[a^2-8a+9]}{h} \\ &= \lim_{h\to 0}\frac{a^2+2ah+h^2-8a-8h+9-a^2+8a-9}{h} \\ &= \lim_{h\to 0}\frac{2ah+h^2-8h}{h} \\ &= \lim_{h\to 0}\frac{h(2a+h-8)}{h} \\ &= \lim_{h\to 0}(2a+h-8) \\ &= 2a+0-8 \\ &= 2a-8 \end{aligned}$$

so

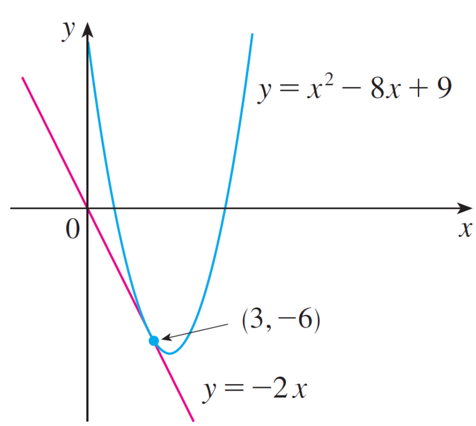
$$f'(a) = 2a - 8$$

To find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$ we will use the point-slope equation of a line:

$$y - y_0 = m(x - x_0)$$

Since $(x_0, y_0) = (3, -6)$ and $m = f'(3) = 2 \cdot 3 - 8 = -2$, we obtain

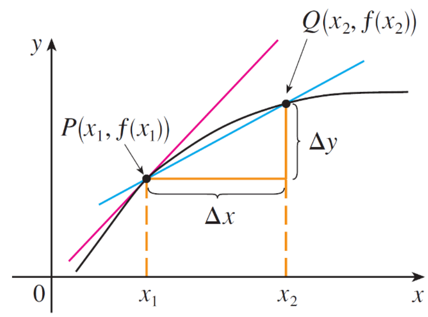
$$y - (-6) = -2(x - 3) \quad \text{or} \quad y = -2x$$



Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$



and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in the figure above.

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the (**instantaneous**) **rate of change of y with respect to x** at $x = x_1$, which is interpreted as the slope of the tangent curve $y = f(x)$ at $(P(x_1, f(x_1))$:

$$\boxed{\text{instantaneous rate of change } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

Since $f'(x_1) = \lim\limits_{x_2 \to x_1} \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$, we have a second interpretation of the rate of change:

> The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$

EXAMPLE: Let $D(t)$ be the US national debt at time t. The table below gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1980 to 2000. Interpret and estimate the value of $D'(1990)$.

t	$D(t)$
1980	930.2
1985	1945.9
1990	3233.3
1995	4974.0
2000	5674.2

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1980	930.2
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Solution: The derivative $D'(1990)$ means the rate of change of D with respect to t when $t = 1990$, that is, the rate of increase of the national debt in 1990. We know that

$$D'(1990) = \lim_{t \to 1990} \frac{D(t) - D(1990)}{t - 1990}$$

So we compute values of the difference quotient (the average rates of change) as follows:

t	$\dfrac{D(t) - D(1990)}{t - 1990}$
1980	230.31
1985	257.48
1995	348.14
2000	244.09

From this table we see that $D'(1990)$ lies somewhere between 257.48 and 348.14 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 1980 and 2000.] We estimate that the rate of increase of the national debt of the United States in 1990 was the average of these two numbers, namely

$$D'(1990) \approx \frac{257.48 + 348.14}{2} = 302.81 \approx 303 \text{ billion dollars per year}$$

Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 1990$.