Limits Involving Infinity

I. Infinite Limits

DEFINITION: The notation

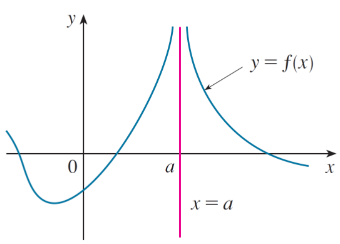
$$\boxed{\lim_{x\to a} f(x) = \infty}$$

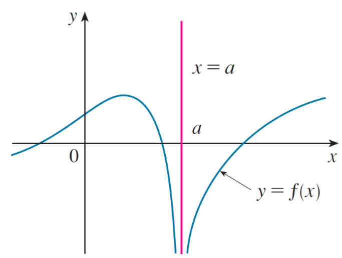
means that the values of $f(x)$ can be made arbitrary large (as large as we like) by taking x sufficiently close to a (on either side of a) but not equal to a.

Similarly,

$$\boxed{\lim_{x\to a} f(x) = -\infty}$$

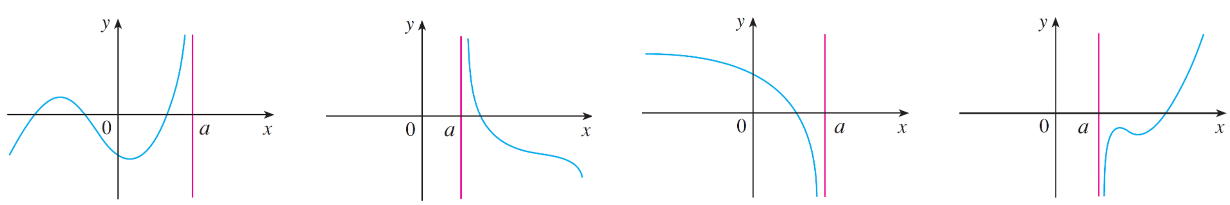
means that the values of $f(x)$ can be made as large negative as we like by taking x sufficiently close to a (on either side of a) but not equal to a.





Similar definitions can be given for the one-sided infinite limits

$$\boxed{\begin{array}{ll} \displaystyle\lim_{x\to a^-} f(x) = \infty & \displaystyle\lim_{x\to a^+} f(x) = \infty \\ \displaystyle\lim_{x\to a^-} f(x) = -\infty & \displaystyle\lim_{x\to a^+} f(x) = -\infty \end{array}}$$



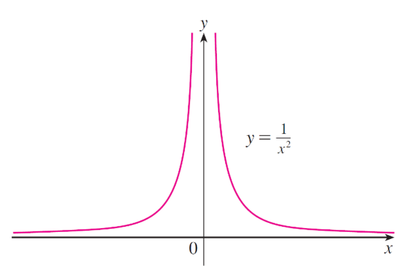
(a) $\lim_{x\to a^-} f(x) = \infty$ (b) $\lim_{x\to a^+} f(x) = \infty$ (c) $\lim_{x\to a^-} f(x) = -\infty$ (d) $\lim_{x\to a^+} f(x) = -\infty$

EXAMPLES:

1. The limits $\displaystyle\lim_{x\to 0}\frac{1}{x^2}$, $\displaystyle\lim_{x\to 0^-}\frac{1}{x^2}$, $\displaystyle\lim_{x\to 0^+}\frac{1}{x^2}$ D.N.E., moreover

$$\lim_{x\to 0}\frac{1}{x^2} = \lim_{x\to 0^-}\frac{1}{x^2} = \lim_{x\to 0^+}\frac{1}{x^2} = \infty$$

x	$f(x)$
$\pm$ 0.1	100
$\pm$ 0.01	10000
$\pm$ 0.001	1000000
$\pm$ 0.0001	100000000
$\pm$ 0.00001	10000000000



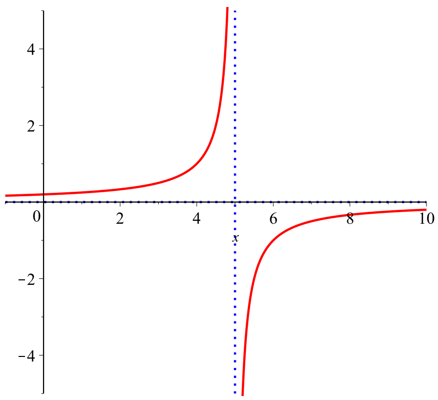
2. The limits $\lim\limits_{x\to 5^-}\dfrac{1}{5-x}$ and $\lim\limits_{x\to 5^+}\dfrac{1}{5-x}$ D.N.E., moreover

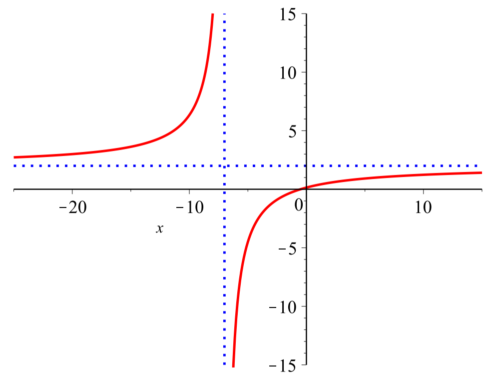
$$\lim_{x\to 5^-}\frac{1}{5-x}=\left[\begin{array}{l}\text{WORK:}\\ \dfrac{1}{5-4.99}=\dfrac{1}{0.01}=\dfrac{+\text{“NOT SMALL”}}{+\text{“SMALL”}}=+\text{“BIG”}\end{array}\right]=\infty$$

and

$$\lim_{x\to 5^+}\frac{1}{5-x}=\left[\begin{array}{l}\text{WORK:}\\ \dfrac{1}{5-5.01}=\dfrac{1}{-0.01}=\dfrac{+\text{“NOT SMALL”}}{-\text{“SMALL”}}=-\text{“BIG”}\end{array}\right]=-\infty$$

Therefore $\lim\limits_{x\to 5}\dfrac{1}{5-x}$ D.N.E. and neither ∞ nor $-\infty$ (see the Figure below (left)).





3. The limits $\lim\limits_{x\to -7^-}\dfrac{2x+1}{x+7}$ and $\lim\limits_{x\to -7^+}\dfrac{2x+1}{x+7}$ D.N.E., moreover

$$\lim_{x\to -7^-}\frac{2x+1}{x+7}=\left[\begin{array}{l}\text{WORK:}\\ \dfrac{2(-7.01)+1}{-7.01+7}\approx\dfrac{-13}{-0.01}=\dfrac{-\text{“NOT SMALL”}}{-\text{“SMALL”}}=+\text{“BIG”}\end{array}\right]=\infty$$

and

$$\lim_{x\to -7^+}\frac{2x+1}{x+7}=\left[\begin{array}{l}\text{WORK:}\\ \dfrac{2(-6.99)+1}{-6.99+7}\approx\dfrac{-13}{0.01}=\dfrac{-\text{“NOT SMALL”}}{+\text{“SMALL”}}=-\text{“BIG”}\end{array}\right]=-\infty$$

Therefore $\lim\limits_{x\to -7}\dfrac{2x+1}{x+7}$ D.N.E. and neither ∞ nor $-\infty$ (see the Figure above (right)).

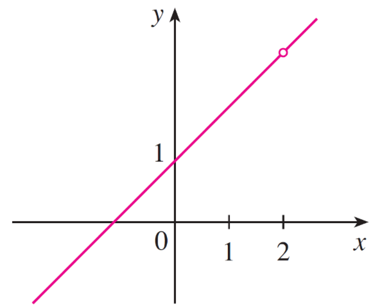
REMARK: Recall that even if the denominator of a function f goes to 0, it does not necessary mean that the limit of f is ∞ or $-\infty$. In fact,

$$\lim_{x\to 2^-}\frac{x^2-x-2}{x-2}=\left[\frac{0}{0}\right]=\lim_{x\to 2^+}\frac{x^2-x-2}{x-2}$$

$$=\lim_{x\to 2}\frac{x^2-x-2}{x-2}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+1)}{x-2}$$

$$=\lim_{x\to 2}(x+1)=3$$



DEFINITION: The line $x=a$ is called a **vertical asymptote** of the curve $y=f(x)$ if at least one of the following statements is true:

$\lim_{x\to a} f(x)=\infty$	$\lim_{x\to a^-} f(x)=\infty$	$\lim_{x\to a^+} f(x)=\infty$
$\lim_{x\to a} f(x)=-\infty$	$\lim_{x\to a^-} f(x)=-\infty$	$\lim_{x\to a^+} f(x)=-\infty$

EXAMPLES:

1. Find all vertical asymptotes of $f(x)=-\dfrac{8}{x^2-4}$.

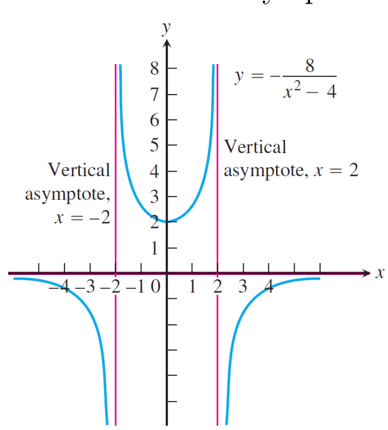
Solution: There are potential vertical asymptotes where $x^2-4=0$, that is where $x=\pm 2$. In fact, we have

$$\lim_{x\to -2^-}\frac{-8}{x^2-4}=\left[\begin{array}{l}\text{WORK:}\\ \dfrac{-8}{(-2.01)^2-4}\approx\dfrac{-8}{4.04-4}=\dfrac{-8}{0.04}=\dfrac{-\text{“NOT SMALL”}}{+\text{“SMALL”}}=-\text{“BIG”}\end{array}\right]=-\infty$$

and

$$\lim_{x\to 2^+}\frac{-8}{x^2-4}=\left[\begin{array}{l}\text{WORK:}\\ \dfrac{-8}{(2.01)^2-4}\approx\dfrac{-8}{4.04-4}=\dfrac{-8}{0.04}=\dfrac{-\text{“NOT SMALL”}}{+\text{“SMALL”}}=-\text{“BIG”}\end{array}\right]=-\infty$$

This shows that the lines $x=\pm 2$ are the vertical asymptotes of f.

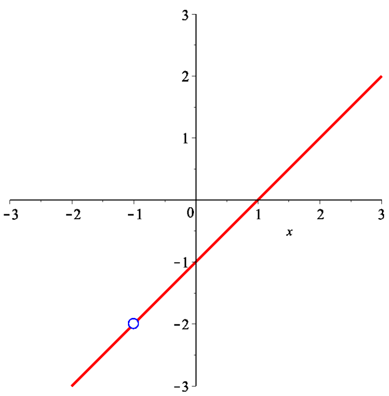


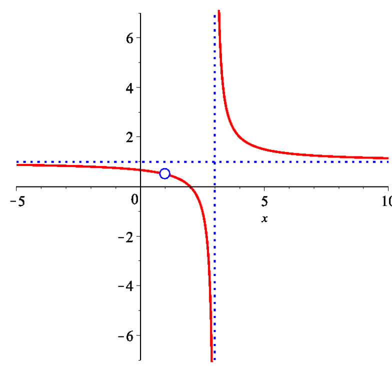
2. Find all vertical asymptotes of $f(x) = \dfrac{x^2-1}{x+1}$.

Solution: There is a potential vertical asymptote where $x+1=0$, that is where $x=-1$. Moreover, if $x=-1$ is a vertical asymptote, it is the only one. However,

$$\lim_{x\to -1}\frac{x^2-1}{x+1} = \lim_{x\to -1}\frac{(x-1)(x+1)}{x+1} = \lim_{x\to -1}(x-1) = -2$$

and therefore $\displaystyle\lim_{x\to -1^{\pm}}\frac{x^2-1}{x+1} = -2$. This shows that f does not have vertical asymptotes (see the Figure below (left)).





3. Find all vertical asymptotes of $f(x) = \dfrac{x^2-3x+2}{x^2-4x+3}$.

Solution: There are potential vertical asymptotes where $x^2-4x+3=0$, that is where $x=1,3$. However,

$$\lim_{x\to 1}\frac{x^2-3x+2}{x^2-4x+3} = \lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x-3)} = \lim_{x\to 1}\frac{x-2}{x-3} = \frac{1}{2}$$

and therefore $\displaystyle\lim_{x\to 1^{\pm}}\frac{x^2-3x+2}{x^2-4x+3} = \frac{1}{2}$. Hence $x=1$ is not a vertical asymptote. On the other hand,

$$\lim_{x\to 3^+}\frac{x^2-3x+2}{x^2-4x+3} = \lim_{x\to 3^+}\frac{(x-1)(x-2)}{(x-1)(x-3)}$$

$$= \lim_{x\to 3^+}\frac{x-2}{x-3} = \left[\begin{array}{l}\text{WORK:}\\ \dfrac{3.01-2}{3.01-3} = \dfrac{1.01}{0.01} = \dfrac{+\text{“NOT SMALL”}}{+\text{“SMALL”}} = +\text{“BIG”}\end{array}\right] = \infty$$

This shows that the line $x=3$ is the only vertical asymptote of f (see the Figure above (right)).

4. Find all vertical asymptotes of $f(x) = \cot 2x$.

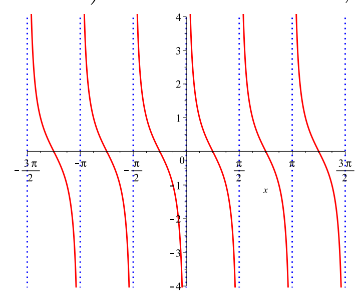
Solution: Because

$$\cot 2x = \frac{\cos 2x}{\sin 2x}$$

there are potential vertical asymptotes where $\sin 2x = 0$. In fact, since $\sin 2x \to 0^+$ as $x \to 0^+$ and $\sin 2x \to 0^-$ as $x \to 0^-$, whereas $\cos 2x$ is positive (and not near 0) when x is near 0, we have

$$\lim_{x\to 0^+} \cot 2x = \infty \quad \text{and} \quad \lim_{x\to 0^-} \cot 2x = -\infty$$

This shows that the line $x = 0$ is a vertical asymptote. Similar reasoning shows that the lines $x = n\pi/2$, where n is an integer, are all vertical asymptotes of $f(x) = \cot 2x$. The graph confirms that.

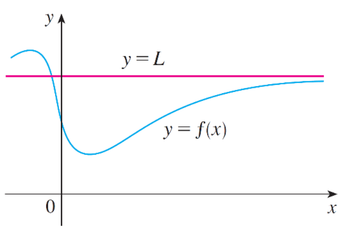


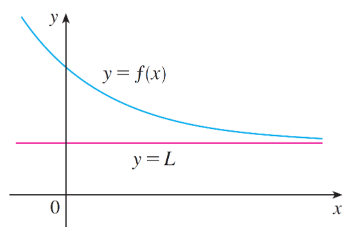
II. Limits at Infinity

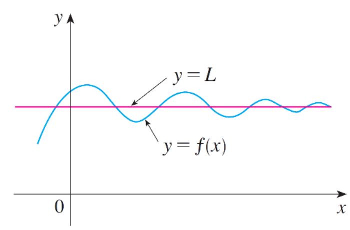
DEFINITION: Let f be a function defined on some interval (a, ∞). Then

$$\boxed{\lim_{x\to\infty} f(x) = L}$$

means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large.



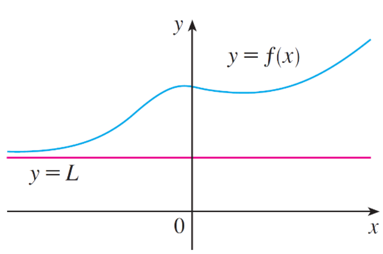


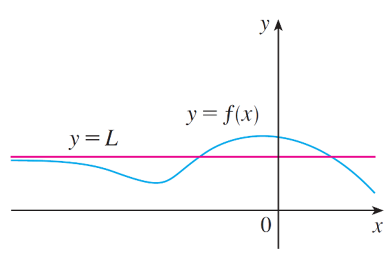


Similarly, the notation

$$\boxed{\lim_{x\to-\infty} f(x) = L}$$

means that the values of $f(x)$ can be made arbitrary close to L by taking x sufficiently large negative.





DEFINITION: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

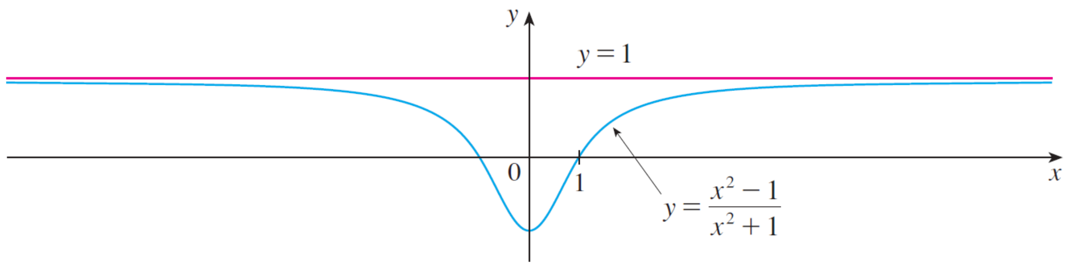
$$\boxed{\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L}$$

EXAMPLE: Find the horizontal asymptote of $f(x)=\dfrac{x^2-1}{x^2+1}$.

Solution: We have

$$\lim_{x\to\pm\infty}\frac{x^2-1}{x^2+1}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\frac{x^2-1}{x^2}}{\frac{x^2+1}{x^2}}\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{\frac{x^2}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}\stackrel{A}{=}\lim_{x\to\pm\infty}\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}\stackrel{C}{=}\frac{1-0}{1+0}=1$$

It follows that $y=1$ is the horizontal asymptote of $f(x)=\dfrac{x^2-1}{x^2+1}$. The graph below confirms that:



EXAMPLE: Find all horizontal asymptotes of $f(x)=\dfrac{x}{\sqrt{x^2+1}}$.

Solution: We have

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\infty}\frac{\frac{x}{\sqrt{x^2}}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}\stackrel{A}{=}\lim_{x\to\infty}\frac{\frac{x}{x}}{\sqrt{\frac{x^2+1}{x^2}}}\stackrel{A}{=}\lim_{x\to\infty}\frac{1}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}$$

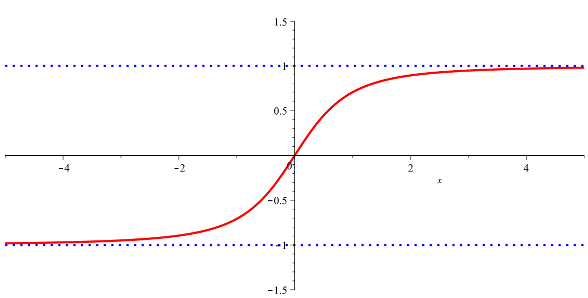
$$\stackrel{A}{=}\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x^2}}}\stackrel{C}{=}\frac{1}{\sqrt{1+0}}=1$$

and

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to-\infty}\frac{\frac{x}{\sqrt{x^2}}}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}\stackrel{A}{=}\lim_{x\to-\infty}\frac{\frac{x}{-x}}{\sqrt{\frac{x^2+1}{x^2}}}\stackrel{A}{=}\lim_{x\to-\infty}\frac{-1}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}$$

$$\stackrel{A}{=}\lim_{x\to-\infty}\frac{-1}{\sqrt{1+\frac{1}{x^2}}}\stackrel{C}{=}\frac{-1}{\sqrt{1+0}}=-1$$

It follows that $y=\pm1$ are the horizontal asymptotes of $f(x)=\dfrac{x}{\sqrt{x^2+1}}$. The graph below confirms that:



EXAMPLES:

1. $$\lim_{x\to\infty}\frac{2x^3-x+5}{x^3+x^2-1}=\left[\frac{\infty}{\infty}\right]\overset{A}{=}\lim_{x\to\infty}\frac{\frac{2x^3-x+5}{x^3}}{\frac{x^3+x^2-1}{x^3}}\overset{A}{=}\lim_{x\to\infty}\frac{\frac{2x^3}{x^3}-\frac{x}{x^3}+\frac{5}{x^3}}{\frac{x^3}{x^3}+\frac{x^2}{x^3}-\frac{1}{x^3}}\overset{A}{=}\lim_{x\to\infty}\frac{2-\frac{1}{x^2}+\frac{5}{x^3}}{1+\frac{1}{x}-\frac{1}{x^3}}$$
$$\overset{C}{=}\frac{2-0+0}{1+0-0}=2$$

2. $$\lim_{x\to\pm\infty}\frac{3x+3x^2-7}{x+1-5x^2}=\left[\frac{\infty}{\infty}\right]\overset{A}{=}\lim_{x\to\pm\infty}\frac{\frac{3x+3x^2-7}{x^2}}{\frac{x+1-5x^2}{x^2}}\overset{A}{=}\lim_{x\to\pm\infty}\frac{\frac{3x}{x^2}+\frac{3x^2}{x^2}-\frac{7}{x^2}}{\frac{x}{x^2}+\frac{1}{x^2}-\frac{5x^2}{x^2}}\overset{A}{=}\lim_{x\to\pm\infty}\frac{\frac{3}{x}+3-\frac{7}{x^2}}{\frac{1}{x}+\frac{1}{x^2}-5}$$
$$\overset{C}{=}\frac{0+3-0}{0+0-5}=-\frac{3}{5}$$

3. $$\lim_{x\to\infty}\frac{4\sqrt[3]{x}-\sqrt[2]{x}-1}{2\sqrt[3]{x}-9\sqrt[2]{x}+1}=\left[\frac{\infty}{\infty}\right]\overset{A}{=}\lim_{x\to\infty}\frac{4x^{1/3}-x^{1/2}-1}{2x^{1/3}-9x^{1/2}+1}\overset{A}{=}\lim_{x\to\infty}\frac{\frac{4x^{1/3}-x^{1/2}-1}{x^{1/2}}}{\frac{2x^{1/3}-9x^{1/2}+1}{x^{1/2}}}$$
$$\overset{A}{=}\lim_{x\to\infty}\frac{\frac{4x^{1/3}}{x^{1/2}}-\frac{x^{1/2}}{x^{1/2}}-\frac{1}{x^{1/2}}}{\frac{2x^{1/3}}{x^{1/2}}-\frac{9x^{1/2}}{x^{1/2}}+\frac{1}{x^{1/2}}}\overset{A}{=}\lim_{x\to\infty}\frac{4x^{1/3-1/2}-1-\frac{1}{x^{1/2}}}{2x^{1/3-1/2}-9+\frac{1}{x^{1/2}}}\overset{A}{=}\lim_{x\to\infty}\frac{4x^{-1/6}-1-\frac{1}{x^{1/2}}}{2x^{-1/6}-9+\frac{1}{x^{1/2}}}$$
$$\overset{A}{=}\lim_{x\to\infty}\frac{\frac{4}{x^{1/6}}-1-\frac{1}{x^{1/2}}}{\frac{2}{x^{1/6}}-9+\frac{1}{x^{1/2}}}\overset{C}{=}\frac{0-1-0}{0-9+0}=\frac{-1}{-9}=\frac{1}{9}$$

4. $$\lim_{x\to-\infty}\frac{7x^2+10x+20}{x^3-10x^2-1}=\left[\frac{\infty}{\infty}\right]\overset{A}{=}\lim_{x\to-\infty}\frac{\frac{7x^2+10x+20}{x^3}}{\frac{x^3-10x^2-1}{x^3}}\overset{A}{=}\lim_{x\to-\infty}\frac{\frac{7x^2}{x^3}+\frac{10x}{x^3}+\frac{20}{x^3}}{\frac{x^3}{x^3}-\frac{10x^2}{x^3}-\frac{1}{x^3}}$$
$$\overset{A}{=}\lim_{x\to-\infty}\frac{\frac{7}{x}+\frac{10}{x^2}+\frac{20}{x^3}}{1-\frac{10}{x}-\frac{1}{x^3}}\overset{C}{=}\frac{0+0+0}{1-0-0}=0$$

or

$$\lim_{x\to-\infty}\frac{7x^2+10x+20}{x^3-10x^2-1}=\left[\frac{\infty}{\infty}\right]\overset{A}{=}\lim_{x\to-\infty}\frac{\frac{7x^2+10x+20}{x^2}}{\frac{x^3-10x^2-1}{x^2}}\overset{A}{=}\lim_{x\to-\infty}\frac{\frac{7x^2}{x^2}+\frac{10x}{x^2}+\frac{20}{x^2}}{\frac{x^3}{x^2}-\frac{10x^2}{x^2}-\frac{1}{x^2}}$$
$$\overset{A}{=}\lim_{x\to-\infty}\frac{7+\frac{10}{x}+\frac{20}{x^2}}{x-10-\frac{1}{x^2}}\overset{C}{=}\lim_{x\to-\infty}\frac{7+0+0}{x-10-0}\overset{C}{=}\lim_{x\to-\infty}\frac{7}{x-10}\overset{C}{=}0$$

5. $$\lim_{x\to-\infty}\frac{11x^5+1}{4-x^4}$$

5. $$\lim_{x\to-\infty}\frac{11x^5+1}{4-x^4}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to-\infty}\frac{\frac{11x^5+1}{x^4}}{\frac{4-x^4}{x^4}}\stackrel{A}{=}\lim_{x\to-\infty}\frac{\frac{11x^5}{x^4}+\frac{1}{x^4}}{\frac{4}{x^4}-\frac{x^4}{x^4}}$$

$$\stackrel{A}{=}\lim_{x\to-\infty}\frac{11x+\frac{1}{x^4}}{\frac{4}{x^4}-1}\stackrel{C}{=}\left[\frac{-\infty}{-1}\right]=\infty \text{ (D.N.E)}$$

or

$$\lim_{x\to-\infty}\frac{11x^5+1}{4-x^4}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to-\infty}\frac{\frac{11x^5+1}{x^5}}{\frac{4-x^4}{x^5}}\stackrel{A}{=}\lim_{x\to-\infty}\frac{\frac{11x^5}{x^5}+\frac{1}{x^5}}{\frac{4}{x^5}-\frac{x^4}{x^5}}$$

$$\stackrel{A}{=}\lim_{x\to-\infty}\frac{11+\frac{1}{x^5}}{\frac{4}{x^5}-\frac{1}{x}}\stackrel{C}{=}\left[\frac{11}{0^+}\right]=\infty \text{ (D.N.E)}$$

6. $$\lim_{x\to\infty}\frac{x^2+2x+5}{\sqrt{2+x+7x^4}}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\infty}\frac{\frac{x^2+2x+5}{\sqrt{x^4}}}{\frac{\sqrt{2+x+7x^4}}{\sqrt{x^4}}}\stackrel{A}{=}\lim_{x\to\infty}\frac{\frac{x^2+2x+5}{x^2}}{\sqrt{\frac{2+x+7x^4}{x^4}}}$$

$$\stackrel{A}{=}\lim_{x\to\infty}\frac{\frac{x^2}{x^2}+\frac{2x}{x^2}+\frac{5}{x^2}}{\sqrt{\frac{2}{x^4}+\frac{x}{x^4}+\frac{7x^4}{x^4}}}\stackrel{A}{=}\lim_{x\to\infty}\frac{1+\frac{2}{x}+\frac{5}{x^2}}{\sqrt{\frac{2}{x^4}+\frac{1}{x^3}+7}}\stackrel{C}{=}\frac{1+0+0}{\sqrt{0+0+7}}=\frac{1}{\sqrt{7}}$$

7. $$\lim_{x\to\infty}\frac{\sqrt{2x^5-1}}{\sqrt{3x^5+2}}=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\infty}\sqrt{\frac{2x^5-1}{3x^5+2}}\stackrel{A}{=}\lim_{x\to\infty}\sqrt{\frac{\frac{2x^5-1}{x^5}}{\frac{3x^5+2}{x^5}}}\stackrel{A}{=}\lim_{x\to\infty}\sqrt{\frac{\frac{2x^5}{x^5}-\frac{1}{x^5}}{\frac{3x^5}{x^5}+\frac{2}{x^5}}}$$

$$\stackrel{A}{=}\lim_{x\to\infty}\sqrt{\frac{2-\frac{1}{x^5}}{3+\frac{2}{x^5}}}\stackrel{C}{=}\sqrt{\frac{2-0}{3+0}}=\sqrt{\frac{2}{3}}$$

REMARK: For more examples of this type, see Appendix I.

8. $$\lim_{x\to\infty}\sin\left(\frac{x+3}{6x^2-5}\right)=\left[\frac{\infty}{\infty}\right]\stackrel{A}{=}\lim_{x\to\infty}\sin\left(\frac{\frac{x+3}{x^2}}{\frac{6x^2-5}{x^2}}\right)\stackrel{A}{=}\lim_{x\to\infty}\sin\left(\frac{\frac{x}{x^2}+\frac{3}{x^2}}{\frac{6x^2}{x^2}-\frac{5}{x^2}}\right)$$

$$\stackrel{A}{=}\lim_{x\to\infty}\sin\left(\frac{\frac{1}{x}+\frac{3}{x^2}}{6-\frac{5}{x^2}}\right)\stackrel{C}{=}\sin\left(\frac{0+0}{6-0}\right)=\sin\left(\frac{0}{6}\right)=\sin 0=0$$

9. $$\lim_{x\to\infty}(\sqrt{x^2+4}-x)=[\infty-\infty]\stackrel{A}{=}\lim_{x\to\infty}\frac{\sqrt{x^2+4}-x}{1}\stackrel{A}{=}\lim_{x\to\infty}\frac{(\sqrt{x^2+4}-x)(\sqrt{x^2+4}+x)}{1\cdot(\sqrt{x^2+4}+x)}$$

$$\stackrel{A}{=}\lim_{x\to\infty}\frac{(\sqrt{x^2+4})^2-x^2}{\sqrt{x^2+4}+x}\stackrel{A}{=}\lim_{x\to\infty}\frac{x^2+4-x^2}{\sqrt{x^2+4}+x}\stackrel{A}{=}\lim_{x\to\infty}\frac{4}{\sqrt{x^2+4}+x}\stackrel{C}{=}0$$

10. $$\lim_{x\to-\infty}(\sqrt{x^2+4}-x)$$

10. $\lim\limits_{x\to-\infty}(\sqrt{x^2+4}-x)=[\infty+\infty]=\infty$ (D.N.E)

REMARK: For more examples of this type, see Appendix II.

11. $\lim\limits_{x\to\infty}(x-\sqrt{x})$

12. $\lim\limits_{x\to\infty}(x^3-x^8)$

13. $\lim\limits_{x\to\infty}(x^3+x^8)$

14. $\lim\limits_{x\to\infty}\sin x$

15. $\lim\limits_{x\to\infty}\dfrac{1+\sin 5x}{\sqrt{1+x}}$

11. $\lim\limits_{x\to\infty}(x-\sqrt{x}) = [\infty-\infty] \stackrel{A}{=} \lim\limits_{x\to\infty}\dfrac{x-\sqrt{x}}{1} \stackrel{A}{=} \lim\limits_{x\to\infty}\dfrac{(x-\sqrt{x})(x+\sqrt{x})}{1\cdot(x+\sqrt{x})} \stackrel{A}{=} \lim\limits_{x\to\infty}\dfrac{x^2-(\sqrt{x})^2}{x+\sqrt{x}}$

$\stackrel{A}{=} \lim\limits_{x\to\infty}\dfrac{x^2-x}{x+\sqrt{x}} \stackrel{A}{=} \lim\limits_{x\to\infty}\dfrac{\frac{x^2-x}{x}}{\frac{x+\sqrt{x}}{x}} \stackrel{A}{=} \lim\limits_{x\to\infty}\dfrac{\frac{x^2}{x}-\frac{x}{x}}{\frac{x}{x}+\frac{\sqrt{x}}{x}} \stackrel{A}{=} \lim\limits_{x\to\infty}\dfrac{x-1}{1+\frac{1}{\sqrt{x}}} \stackrel{C}{=} \left[\dfrac{\infty}{1}\right] = \infty$ (D.N.E)

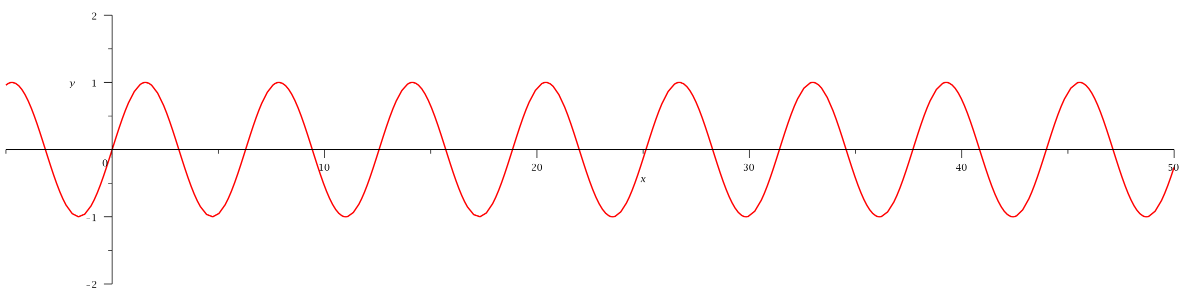
or

$$\lim_{x\to\infty}(x-\sqrt{x}) = [\infty-\infty] \stackrel{A}{=} \lim_{x\to\infty}(\sqrt{x}\cdot\sqrt{x}-1\cdot\sqrt{x}) \stackrel{A}{=} \lim_{x\to\infty}[\sqrt{x}(\sqrt{x}-1)] \stackrel{C}{=} [\infty\cdot\infty] = \infty$$
(D.N.E)

12. $\lim\limits_{x\to\infty}(x^3-x^8) = [\infty-\infty] \stackrel{A}{=} \lim\limits_{x\to\infty} x^3(1-x^5) \stackrel{C}{=} [\infty\cdot(-\infty)] = -\infty$ (D.N.E)

13. $\lim\limits_{x\to\infty}(x^3+x^8) = [\infty+\infty] = \infty$ (D.N.E)

14. $\lim\limits_{x\to\infty}\sin x$ D.N.E



15. $\lim\limits_{x\to\infty}\dfrac{1+\sin 5x}{\sqrt{1+x}} = 0$

Solution: We first note that

$$0 \le 1+\sin 5x \le 2$$

Dividing all three parts of this inequality by $\sqrt{1+x}$, we get

$$0 \le \frac{1+\sin 5x}{\sqrt{1+x}} \le \frac{2}{\sqrt{1+x}}$$

Since

$$\lim_{x\to\infty}\frac{2}{\sqrt{1+x}} = 0$$

it follows that

$$\lim_{x\to\infty}\frac{1+\sin 5x}{\sqrt{1+x}} = 0$$

by the Squeeze Theorem.

Appendix I

1. Find $\lim_{x\to\infty}\frac{\sqrt[3]{x+5}}{\sqrt{x+7}}$.

Solution: We have

$$\lim_{x\to\infty}\frac{\sqrt[3]{x+5}}{\sqrt{x+7}}=\left[\frac{\infty}{\infty}\right]\overset{A}{=}\lim_{x\to\infty}\frac{\frac{\sqrt[3]{x+5}}{\sqrt{x}}}{\frac{\sqrt{x+7}}{\sqrt{x}}}=\left\{\sqrt{x}=x^{\frac{1}{2}}=x^{\frac{3}{2}\cdot\frac{1}{3}}=\left(x^{\frac{3}{2}}\right)^{\frac{1}{3}}=\sqrt[3]{x^{3/2}}\right\}\overset{A}{=}\lim_{x\to\infty}\frac{\frac{\sqrt[3]{x+5}}{\sqrt[3]{x^{3/2}}}}{\frac{\sqrt{x+7}}{\sqrt{x}}}$$

$$\overset{A}{=}\lim_{x\to\infty}\frac{\sqrt[3]{\frac{x+5}{x^{3/2}}}}{\sqrt{\frac{x+7}{x}}}\overset{A}{=}\lim_{x\to\infty}\frac{\sqrt[3]{\frac{x}{x^{3/2}}+\frac{5}{x^{3/2}}}}{\sqrt{\frac{x}{x}+\frac{7}{x}}}\overset{A}{=}\lim_{x\to\infty}\frac{\sqrt[3]{\frac{1}{x^{1/2}}+\frac{5}{x^{3/2}}}}{\sqrt{1+\frac{7}{x}}}\overset{C}{=}\frac{\sqrt[3]{0+0}}{\sqrt{1+0}}=\frac{0}{1}=0$$

2. Find $\lim_{x\to\infty}\frac{\sqrt[6]{3x^2+4}}{\sqrt[9]{1-2x^3}}$.

Solution: We have

$$\lim_{x\to\infty}\frac{\sqrt[6]{3x^2+4}}{\sqrt[9]{1-2x^3}}=\left[\frac{\infty}{\infty}\right]\overset{A}{=}\lim_{x\to\infty}\frac{\frac{\sqrt[6]{3x^2+4}}{\sqrt[9]{x^3}}}{\frac{\sqrt[9]{1-2x^3}}{\sqrt[9]{x^3}}}=\left\{\sqrt[9]{x^3}=x^{\frac{3}{9}}=x^{\frac{1}{3}}=x^{2\cdot\frac{1}{6}}=\left(x^2\right)^{\frac{1}{6}}=\sqrt[6]{x^2}\right\}\overset{A}{=}\lim_{x\to\infty}\frac{\frac{\sqrt[6]{3x^2+4}}{\sqrt[6]{x^2}}}{\frac{\sqrt[9]{1-2x^3}}{\sqrt[9]{x^3}}}$$

$$\overset{A}{=}\lim_{x\to\infty}\frac{\sqrt[6]{\frac{3x^2+4}{x^2}}}{\sqrt[9]{\frac{1-2x^3}{x^3}}}\overset{A}{=}\lim_{x\to\infty}\frac{\sqrt[6]{\frac{3x^2}{x^2}+\frac{4}{x^2}}}{\sqrt[9]{\frac{1}{x^3}-\frac{2x^3}{x^3}}}\overset{A}{=}\lim_{x\to\infty}\frac{\sqrt[6]{3+\frac{4}{x^2}}}{\sqrt[9]{\frac{1}{x^3}-2}}\overset{C}{=}\frac{\sqrt[6]{3+0}}{\sqrt[9]{0-2}}=-\frac{\sqrt[6]{3}}{\sqrt[9]{2}}$$

Appendix II

1. Find $\lim\limits_{x\to\infty}[x(\sqrt{x^2+4}-x)]$.

Solution: Since

$$\lim_{x\to\infty}(\sqrt{x^2+4}-x)=0$$

by Example 9 from page 8, it follows that $\lim\limits_{x\to\infty}[x(\sqrt{x^2+4}-x)]$ is $\infty\cdot 0$ type of an indeterminate form. We have

$$\begin{aligned}\lim_{x\to\infty}[x(\sqrt{x^2+4}-x)] = [\infty\cdot 0] &\overset{A}{=} \lim_{x\to\infty}\frac{x(\sqrt{x^2+4}-x)}{1} \overset{A}{=} \lim_{x\to\infty}\frac{x(\sqrt{x^2+4}-x)(\sqrt{x^2+4}+x)}{1\cdot(\sqrt{x^2+4}+x)}\\ &\overset{A}{=} \lim_{x\to\infty}\frac{x[(\sqrt{x^2+4})^2-x^2]}{\sqrt{x^2+4}+x} \overset{A}{=} \lim_{x\to\infty}\frac{x[x^2+4-x^2]}{\sqrt{x^2+4}+x}\\ &\overset{A}{=} \lim_{x\to\infty}\frac{4x}{\sqrt{x^2+4}+x} = \left[\frac{\infty}{\infty}\right] \overset{A}{=} \lim_{x\to\infty}\frac{\frac{4x}{x}}{\frac{\sqrt{x^2+4}+x}{x}} \overset{A}{=} \lim_{x\to\infty}\frac{4}{\frac{\sqrt{x^2+4}}{x}+\frac{x}{x}}\\ &\overset{A}{=} \lim_{x\to\infty}\frac{4}{\frac{\sqrt{x^2+4}}{\sqrt{x^2}}+1} \overset{A}{=} \lim_{x\to\infty}\frac{4}{\sqrt{\frac{x^2+4}{x^2}}+1} \overset{A}{=} \lim_{x\to\infty}\frac{4}{\sqrt{\frac{x^2}{x^2}+\frac{4}{x^2}}+1}\\ &\overset{A}{=} \lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{4}{x^2}}+1} \overset{C}{=} \frac{4}{\sqrt{1+0}+1} = 2\end{aligned}$$

2. Find $\lim\limits_{x\to-\infty}[x(\sqrt{x^2+4}+x)]$.

Solution: We have

$$\begin{aligned}\lim_{x\to-\infty}[x(\sqrt{x^2+4}+x)] = [\infty\cdot 0] &\overset{A}{=} \lim_{x\to-\infty}\frac{x(\sqrt{x^2+4}+x)}{1} \overset{A}{=} \lim_{x\to-\infty}\frac{x(\sqrt{x^2+4}+x)(\sqrt{x^2+4}-x)}{1\cdot(\sqrt{x^2+4}-x)}\\ &\overset{A}{=} \lim_{x\to-\infty}\frac{x[(\sqrt{x^2+4})^2-x^2]}{\sqrt{x^2+4}-x} \overset{A}{=} \lim_{x\to-\infty}\frac{x[x^2+4-x^2]}{\sqrt{x^2+4}-x}\\ &\overset{A}{=} \lim_{x\to-\infty}\frac{4x}{\sqrt{x^2+4}-x} = \left[\frac{\infty}{\infty}\right] \overset{A}{=} \lim_{x\to-\infty}\frac{\frac{4x}{x}}{\frac{\sqrt{x^2+4}-x}{x}} \overset{A}{=} \lim_{x\to-\infty}\frac{4}{\frac{\sqrt{x^2+4}}{x}-\frac{x}{x}}\\ &\overset{A}{=} \lim_{x\to-\infty}\frac{4}{\frac{\sqrt{x^2+4}}{-\sqrt{x^2}}-1} \overset{A}{=} \lim_{x\to-\infty}\frac{4}{-\sqrt{\frac{x^2+4}{x^2}}-1} \overset{A}{=} \lim_{x\to-\infty}\frac{4}{-\sqrt{\frac{x^2}{x^2}+\frac{4}{x^2}}-1}\\ &\overset{A}{=} \lim_{x\to-\infty}\frac{4}{-\sqrt{1+\frac{4}{x^2}}-1} \overset{C}{=} \frac{4}{-\sqrt{1+0}-1} = -2\end{aligned}$$