Continuity

DEFINITION 1: A function \( f \) is \textbf{continuous at a number} \( a \) if
\[
\lim_{{x \to a}} f(x) = f(a)
\]

REMARK: It follows from the definition that \( f \) is continuous at \( a \) if and only if
1. \( f(a) \) is defined.
2. \( \lim_{{x \to a^-}} f(x) \) and \( \lim_{{x \to a^+}} f(x) \) exist.
3. \( \lim_{{x \to a^-}} f(x) = \lim_{{x \to a^+}} f(x) = f(a) \).

REMARK: The discontinuities in parts (b) and (c) are called \textbf{removable discontinuities} because we could remove them by redefining \( f \) at just the single number 0. The discontinuity in part (d) is called \textbf{jump discontinuity} because the function “jumps” from one value to another. The discontinuities in parts (e) and (f) are called \textbf{infinite} or \textbf{essential discontinuities}.

EXAMPLE:
(a) The function
\[ f(x) = \frac{1}{1-x^2} \]
is discontinuous at \( x = \pm 1 \), since \( f(x) \) is not defined at these points.

(b) The function
\[ f(x) = \begin{cases} 
2x - 1 & \text{if } x \leq 2 \\
x^2 & \text{if } x > 2
\end{cases} \]
is discontinuous at \( x = 2 \). In fact,
\[
\lim_{{x \to 2^-}} f(x) = \lim_{{x \to 2^-}} (2x - 1) = 2 \cdot 2 - 1 = 3 \quad \text{and} \quad \lim_{{x \to 2^+}} f(x) = \lim_{{x \to 2^+}} x^2 = 2^2 = 4
\]
so \( \lim_{{x \to 2^-}} f(x) \neq \lim_{{x \to 2^+}} f(x) \), therefore \( f(x) \) is discontinuous at \( x = 2 \).

EXAMPLE: Where are each of the following functions discontinuous?

(a) \( f(x) = \frac{x^2 - x - 2}{x - 2} \)
(b) \( f(x) = \begin{cases} 
\frac{1}{x^2} & \text{if } x \neq 0 \\
1 & \text{if } x = 0
\end{cases} \)
(c) \( f(x) = \begin{cases} 
\frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\
1 & \text{if } x = 2
\end{cases} \)
EXAMPLE: Where are each of the following functions discontinuous?

(a) \( f(x) = \frac{x^2 - x - 2}{x - 2} \) 

(b) \( f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

(c) \( f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \)

Solution:

(a) The function

\[ f(x) = \frac{x^2 - x - 2}{x - 2} \]

is discontinuous at \( x = 2 \), since \( f(x) \) is not defined at this point.

(b) The function

\[ f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \]

is discontinuous at \( x = 0 \), since \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x^2} \) does not exist.

(c) The function

\[ f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \]

is discontinuous at \( x = 2 \), since

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2^-} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2^-} (x + 1) = 3 \]

which is not equal to \( f(2) = 1 \).
DEFINITION 2: A function $f$ is **continuous from the right at a number** $a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

and $f$ is **continuous from the left at** $a$ if

$$\lim_{x \to a^-} f(x) = f(a)$$

EXAMPLE: The graph of a function $g$ is shown.

(a) At which points $a$ in \{0, 1, 2, 3, 4, 5\} is $g$ continuous?
(b) At which points $a$ in \{0, 1, 2, 3, 4, 5\} is $g$ continuous from the right?
(c) At which points $a$ in \{0, 1, 2, 3, 4, 5\} is $g$ continuous from the left?
EXAMPLE: The graph of a function $g$ is shown.

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(c) At which points $a$ in $\{0, 1, 2, 3, 4, 5\}$ is $g$ continuous from the left?

Solution:

(a) The function $g$ is continuous at $a = 0, 2, 5$. In fact,

(i) The function $g$ is continuous at $a = 0$, since $\lim_{x \to 0^+} g(x) = g(0) = -1$.

(ii) The function $g$ is not continuous at $a = 1$, since $\lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x)$.

(iii) The function $g$ is continuous at $a = 2$, since $\lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2) = 0$.

(iv) The function $g$ is not continuous at $a = 3$, since $\lim_{x \to 3^-} g(x) \neq \lim_{x \to 3^+} g(x)$.

(v) The function $g$ is not continuous at $a = 4$, since $g(4)$ does not exist.

(vi) The function $g$ is continuous at $a = 5$, since $\lim_{x \to 5^-} g(x) = g(5) = 1$.

(b) The function $g$ is continuous from the right at $a = 0, 1, 2, 3$. In fact,

(i) The function $g$ is continuous from the right at $a = 0$, since $\lim_{x \to 0^+} g(x) = g(0)$.

(ii) The function $g$ is continuous from the right at $a = 1$, since $\lim_{x \to 1^+} g(x) = g(1)$.

(iii) The function $g$ is continuous from the right at $a = 2$, since $\lim_{x \to 2^+} g(x) = g(2)$.

(iv) The function $g$ is continuous from the right at $a = 3$, since $\lim_{x \to 3^+} g(x) = g(3)$.

(v) The function $g$ is not continuous from the right at $a = 4$, since $g(4)$ does not exist.

(vi) The function $g$ is not continuous from the right at $a = 5$, since $\lim_{x \to 5^+} g(x)$ does not exist.

(c) The function $g$ is continuous from the left at $a = 2, 5$. In fact,

(i) The function $g$ is not continuous from the left at $a = 0$, since $\lim_{x \to 0^-} g(x)$ does not exist.

(ii) The function $g$ is not continuous from the left at $a = 1$, since $\lim_{x \to 1^-} g(x) \neq g(1)$.

(iii) The function $g$ is continuous from the left at $a = 2$, since $\lim_{x \to 2^-} g(x) = g(2)$.

(iv) The function $g$ is not continuous from the left at $a = 3$, since $\lim_{x \to 3^-} g(x) \neq g(3)$.

(v) The function $g$ is not continuous from the left at $a = 4$, since $g(4)$ does not exist.

(vi) The function $g$ is continuous from the left at $a = 5$, since $\lim_{x \to 5^-} g(x) = g(5)$. 

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DEFINITION 3: A function \( f \) is **continuous on an interval** if it is continuous at every point in the interval. (If \( f \) is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

EXAMPLE: Show that the function \( f(x) = \sqrt{16 - x^4} \) is continuous on the interval \([-2, 2]\).

Solution: If \(-2 < a < 2\), then using the Limit Laws, we have
\[
\lim_{x \to a} f(x) = \lim_{x \to a} \sqrt{16 - x^4} \overset{LL}{=} \sqrt{\lim_{x \to a} (16 - x^4)} \overset{DSP}{=} \sqrt{16 - a^4} = f(a)
\]
therefore by Definition 1 the function is continuous. Similarly, since
\[
\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \sqrt{16 - x^4} \overset{LL}{=} \sqrt{\lim_{x \to -2^+} (16 - x^4)} \overset{DSP}{=} \sqrt{16 - (-2)^4} = 0 = f(-2)
\]
and
\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{16 - x^4} \overset{LL}{=} \sqrt{\lim_{x \to 2^-} (16 - x^4)} \overset{DSP}{=} \sqrt{16 - 2^4} = 0 = f(2)
\]
it follows that \( f \) is continuous from the right at \(-2\) and continuous from the left at 2. Therefore, according to Definition 3, \( f \) is continuous on \([-2, 2]\).

THEOREM: If \( f \) and \( g \) are continuous at \( a \) and \( c \) is a constant, then the following functions are also continuous at \( a \):
\[
\begin{align*}
&cf, \quad f \pm g, \quad fg, \quad f \div g \quad \text{(if } g(a) \neq 0) \quad \overset{\text{the box}}{\text{results}}
\end{align*}
\]

THEOREM:
(a) Any polynomial is continuous everywhere.
(b) Any rational function is continuous wherever it is defined.

In general, the following is true:

THEOREM: The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions.

EXAMPLES:

1. \( f(x) = 17x^9 + 5x^2 + x - 22 \) is continuous on \(( -\infty, \infty)\).
2. \( f(x) = \frac{x + 1}{x - 2} \) is continuous on \(( -\infty, 2) \cup (2, \infty)\).
3. \( f(x) = \frac{x}{x} \) is continuous on \(( -\infty, 0) \cup (0, \infty)\).
4. \( f(x) = 1 \) is continuous on \(( -\infty, \infty)\).
5. \( f(x) = \frac{7x^5 + x - 2}{x^2 - 4} \) is continuous on \(( -\infty, -2) \cup (-2, 2) \cup (2, \infty)\).
6. \( f(x) = \sin x + \sqrt{x} - \frac{1}{x - 4} \) is continuous on \([0, 4) \cup (4, \infty)\).
THEOREM: If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

THEOREM: If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at $a$.

EXAMPLES:

1. The function $f(x) = \cos(x^2 + 1)$ is continuous on $(-\infty, \infty)$ by the above Theorem, because $x^2 + 1$ is continuous on $(-\infty, \infty)$ and $\cos x$ is continuous on $(-\infty, \infty)$.

2. The function $f(x) = \sqrt{16 - x^4}$ is continuous on $[-2, 2]$ by the above Theorem, because $16 - x^4$ is continuous on $(-\infty, \infty)$, $\sqrt{x}$ is continuous on $[0, \infty)$ and $16 - x^4 \geq 0$ on $[-2, 2]$.

THE INTERMEDIATE VALUE THEOREM: Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$.

EXAMPLE: Show that there is a root of the equation $3x^7 - 2x^5 + x - 1 = 0$ between 0 and 1.
Solution: Put $f(x) = 3x^7 - 2x^5 + x - 1$. One can check that

$$f(0) < 0 \quad \text{and} \quad f(1) > 0$$

From this by the IVT it follows that there exists a number $c$ in $(0, 1)$ such that $f(c) = 0$ since $f(x)$ is continuous (polynomial) and 0 is between $f(0)$ and $f(1)$.

EXAMPLE: Show that there is a root of the equation $x^9 + x = 5$.
Solution 1: Put $f(x) = x^9 + x$. One can check that

$$f(1) < 5 \quad \text{and} \quad f(2) > 5$$

From this by the IVT it follows that there exists a number $c$ in $(1, 2)$ such that $f(c) = 5$ since $f(x)$ is continuous (polynomial) and 5 is between $f(1)$ and $f(2)$.

Solution 2: Put $f(x) = x^9 + x - 5$. One can check that

$$f(1) < 0 \quad \text{and} \quad f(2) > 0$$

From this by the IVT it follows that there exists a number $c$ in $(1, 2)$ such that $f(c) = 0$ since $f(x)$ is continuous (polynomial) and 0 is between $f(1)$ and $f(2)$.