Continuity

DEFINITION 1: A function f is **continuous at a number** a if

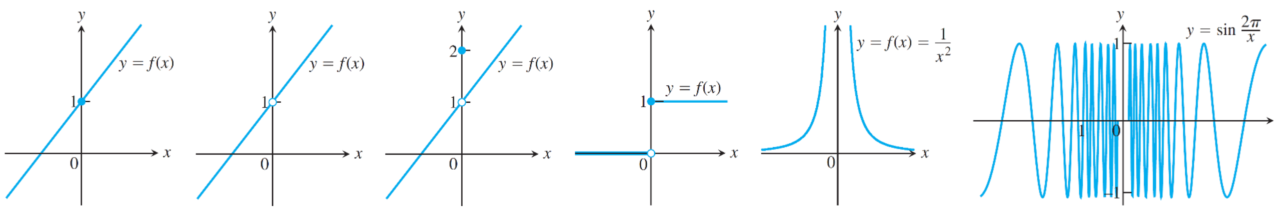
$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

REMARK: It follows from the definition that f is continuous at a if and only if

1. $f(a)$ is defined.

2. $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ exist.

3. $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a)$.



(a) (b) (c) (d) (e) (f)

REMARK: The discontinuities in parts (b) and (c) are called **removable discontinuities** because we could remove them by redefining f at just the single number 0. The discontinuity in part (d) is called **jump discontinuity** because the function "jumps" from one value to another. The discontinuities in parts (e) and (f) are called **infinite** or **essential discontinuities**.

EXAMPLE:

(a) The function

$$f(x) = \frac{1}{1-x^2}$$

is discontinuous at $x = \pm 1$, since $f(x)$ is not defined at these points.

(b) The function

$$f(x) = \begin{cases} 2x-1 & \text{if } x \le 2 \\ x^2 & \text{if } x > 2 \end{cases}$$

is discontinuous at $x = 2$. In fact,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x-1) = 2\cdot 2 - 1 = 3 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 2^2 = 4$$

so $\lim\limits_{x\to 2^-} f(x) \neq \lim\limits_{x\to 2^+} f(x)$, therefore $f(x)$ is discontinuous at $x = 2$.

EXAMPLE: Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2-x-2}{x-2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

EXAMPLE: Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Solution:

(a) The function

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

is discontinuous at $x = 2$, since $f(x)$ is not defined at this point.

(b) The function

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is discontinuous at $x = 0$, since $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{1}{x^2}$ does not exist.

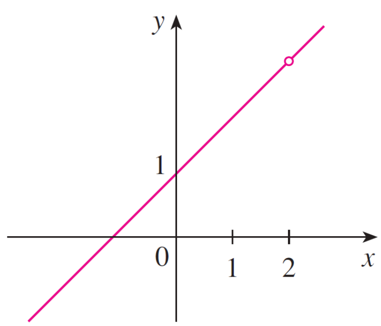
(c) The function

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

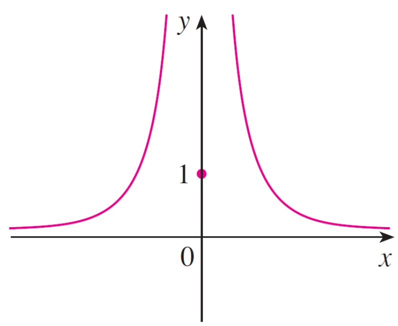
is discontinuous at $x = 2$, since

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2^-} \frac{(x-2)(x+1)}{x - 2} = \lim_{x \to 2^-} (x+1) = 3$$

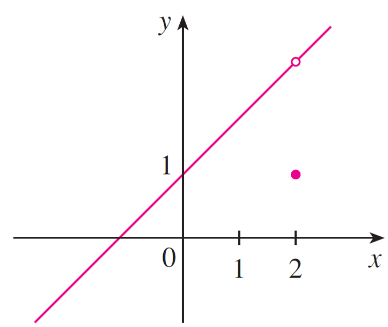
which is not equal to $f(2) = 1$.



(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$



(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

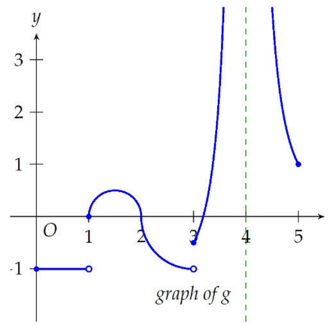
DEFINITION 2: A function f is **continuous from the right at a number a** if

$$\boxed{\lim_{x\to a^+} f(x)=f(a)}$$

and f is **continuous from the left at a** if

$$\boxed{\lim_{x\to a^-} f(x)=f(a)}$$

EXAMPLE: The graph of a function g is shown.

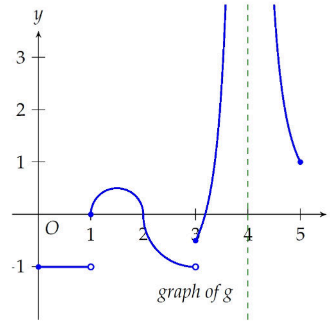


(a) At which points a in $\{0,1,2,3,4,5\}$ is g continuous?

(b) At which points a in $\{0,1,2,3,4,5\}$ is g continuous from the right?

(c) At which points a in $\{0,1,2,3,4,5\}$ is g continuous from the left?

EXAMPLE: The graph of a function g is shown.



(a) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous?

(b) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous from the right?

(c) At which points a in $\{0, 1, 2, 3, 4, 5\}$ is g continuous from the left?

Solution:

(a) The function g is continuous at $a = 0, 2, 5$. In fact,

(i) The function g is continuous at $a = 0$, since $\lim\limits_{x\to 0^+} g(x) = g(0) = -1$.

(ii) The function g is not continuous at $a = 1$, since $\lim\limits_{x\to 1^-} g(x) \neq \lim\limits_{x\to 1^+} g(x)$.

(iii) The function g is continuous at $a = 2$, since $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^+} g(x) = g(2) = 0$.

(iv) The function g is not continuous at $a = 3$, since $\lim\limits_{x\to 3^-} g(x) \neq \lim\limits_{x\to 3^+} g(x)$.

(v) The function g is not continuous at $a = 4$, since $g(4)$ does not exist.

(vi) The function g is continuous at $a = 5$, since $\lim\limits_{x\to 5^-} g(x) = g(5) = 1$.

(b) The function g is continuous from the right at $a = 0, 1, 2, 3$. In fact,

(i) The function g is continuous from the right at $a = 0$, since $\lim\limits_{x\to 0^+} g(x) = g(0)$.

(ii) The function g is continuous from the right at $a = 1$, since $\lim\limits_{x\to 1^+} g(x) = g(1)$.

(iii) The function g is continuous from the right at $a = 2$, since $\lim\limits_{x\to 2^+} g(x) = g(2)$.

(iv) The function g is continuous from the right at $a = 3$, since $\lim\limits_{x\to 3^+} g(x) = g(3)$.

(v) The function g is not continuous from the right at $a = 4$, since $g(4)$ does not exist.

(vi) The function g is not continuous from the right at $a = 5$, since $\lim\limits_{x\to 5^+} g(x)$ does not exist.

(c) The function g is continuous from the left at $a = 2, 5$. In fact,

(i) The function g is not continuous from the left at $a = 0$, since $\lim\limits_{x\to 0^-} g(x)$ does not exist.

(ii) The function g is not continuous from the left at $a = 1$, since $\lim\limits_{x\to 1^-} g(x) \neq g(1)$.

(iii) The function g is continuous from the left at $a = 2$, since $\lim\limits_{x\to 2^-} g(x) = g(2)$.

(iv) The function g is not continuous from the left at $a = 3$, since $\lim\limits_{x\to 3^-} g(x) \neq g(3)$.

(v) The function g is not continuous from the left at $a = 4$, since $g(4)$ does not exist.

(vi) The function g is continuous from the left at $a = 5$, since $\lim\limits_{x\to 5^-} g(x) = g(5)$.

DEFINITION 3: A function f is **continuous on an interval** if it is continuous at every point in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

EXAMPLE: Show that the function $f(x)=\sqrt{16-x^4}$ is continuous on the interval $[-2,2]$.

Solution: If $-2<a<2$, then using the Limit Laws, we have

$$\lim_{x\to a} f(x)=\lim_{x\to a}\sqrt{16-x^4}\overset{LL}{=}\sqrt{\lim_{x\to a}(16-x^4)}\overset{DSP}{=}\sqrt{16-a^4}=f(a)$$

therefore by Definition 1 the function is continuous. Similarly, since

$$\lim_{x\to -2^+} f(x)=\lim_{x\to -2^+}\sqrt{16-x^4}\overset{LL}{=}\sqrt{\lim_{x\to -2^+}(16-x^4)}\overset{DSP}{=}\sqrt{16-(-2)^4}=0=f(-2)$$

and

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\sqrt{16-x^4}\overset{LL}{=}\sqrt{\lim_{x\to 2^-}(16-x^4)}\overset{DSP}{=}\sqrt{16-2^4}=0=f(2)$$

it follows that f is continuous from the right at -2 and continuous from the left at 2. Therefore, according to Definition 3, f is continuous on $[-2,2]$.

THEOREM: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

$$\boxed{cf,\quad f\pm g,\quad fg,\quad \frac{f}{g}\ (\text{if } g(a)\neq 0)}$$

THEOREM:

(a) Any polynomial is continuous everywhere.

(b) Any rational function is continuous wherever it is defined.

In general, the following is true:

THEOREM: The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions.

EXAMPLES:

1. $f(x)=17x^9+5x^2+x-22$ is continuous on $(-\infty,\infty)$.

2. $f(x)=\dfrac{x+1}{x-2}$ is continuous on $(-\infty,2)\cup(2,\infty)$.

3. $f(x)=\dfrac{x}{x}$ is continuous on $(-\infty,0)\cup(0,\infty)$.

4. $f(x)=1$ is continuous on $(-\infty,\infty)$.

5. $f(x)=\dfrac{7x^5+x-2}{x^2-4}$ is continuous on $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$.

6. $f(x)=\sin x+\sqrt{x}-\dfrac{1}{x-4}$ is continuous on $[0,4)\cup(4,\infty)$.

THEOREM: If f is continuous at b and $\lim\limits_{x\to a} g(x) = b$, then $\lim\limits_{x\to a} f(g(x)) = f(b)$. In other words,

$$\boxed{\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)}$$

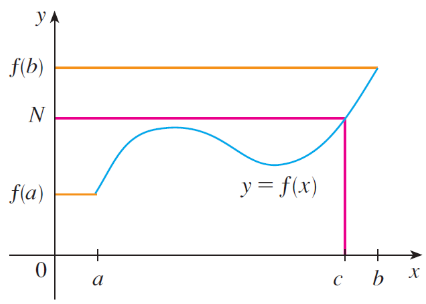
THEOREM: If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

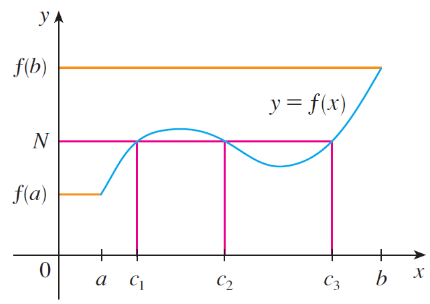
EXAMPLES:

1. The function $f(x) = \cos(x^2+1)$ is continuous on $(-\infty, \infty)$ by the above Theorem, because x^2+1 is continuous on $(-\infty, \infty)$ and $\cos x$ is continuous on $(-\infty, \infty)$.

2. The function $f(x) = \sqrt{16 - x^4}$ is continuous on $[-2, 2]$ by the above Theorem, because $16 - x^4$ is continuous on $(-\infty, \infty)$, $\sqrt{x}$ is continuous on $[0, \infty)$ and $16 - x^4 \geq 0$ on $[-2, 2]$.

THE INTERMEDIATE VALUE THEOREM: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.





EXAMPLE: Show that there is a root of the equation $3x^7 - 2x^5 + x - 1 = 0$ between 0 and 1.

Solution: Put $f(x) = 3x^7 - 2x^5 + x - 1$. One can check that

$$f(0) < 0 \quad \text{and} \quad f(1) > 0$$

From this by the IVT it follows that there exists a number c in $(0, 1)$ such that $f(c) = 0$ since $f(x)$ is continuous (polynomial) and 0 is between $f(0)$ and $f(1)$.

EXAMPLE: Show that there is a root of the equation $x^9 + x = 5$.

Solution 1: Put $f(x) = x^9 + x$. One can check that

$$f(1) < 5 \quad \text{and} \quad f(2) > 5$$

From this by the IVT it follows that there exists a number c in $(1, 2)$ such that $f(c) = 5$ since $f(x)$ is continuous (polynomial) and 5 is between $f(1)$ and $f(2)$.

Solution 2: Put $f(x) = x^9 + x - 5$. One can check that

$$f(1) < 0 \quad \text{and} \quad f(2) > 0$$

From this by the IVT it follows that there exists a number c in $(1, 2)$ such that $f(c) = 0$ since $f(x)$ is continuous (polynomial) and 0 is between $f(1)$ and $f(2)$.