Continuity

DEFINITION 1: A function \( f \) is continuous at a number \( a \) if

\[
\lim_{x \to a} f(x) = f(a)
\]

REMARK: It follows from the definition that \( f \) is continuous at \( a \) if and only if

1. \( f(a) \) is defined.
2. \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) exist.
3. \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a) \).

REMARK: The discontinuities in parts (b) and (c) are called **removable discontinuities** because we could remove them by redefining \( f \) at just the single number 0. The discontinuity in part (d) is called **jump discontinuity** because the function “jumps” from one value to another. The discontinuities in parts (e) and (f) are called **infinite** or **essential discontinuities**.

EXAMPLE:

(a) The function

\[
f(x) = \frac{1}{1 - x^2}
\]

is discontinuous at \( x = \pm 1 \), since \( f(x) \) is not defined at these points.

(b) The function

\[
f(x) = \begin{cases} 
    2x - 1 & \text{if } x \leq 2 \\
    x^2 & \text{if } x > 2 
\end{cases}
\]

is discontinuous at \( x = 2 \). In fact,

\[
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1) = 2 \cdot 2 - 1 = 3 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 2^2 = 4
\]

so \( \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \), therefore \( f(x) \) is discontinuous at \( x = 2 \).

EXAMPLE: Where are each of the following functions discontinuous?

(a) \( f(x) = \frac{x^2 - x - 2}{x - 2} \)  
(b) \( f(x) = \begin{cases} 
    \frac{1}{x^2} & \text{if } x \neq 0 \\
    1 & \text{if } x = 0 
\end{cases} \)  
(c) \( f(x) = \begin{cases} 
    \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\
    1 & \text{if } x = 2 
\end{cases} \)
EXAMPLE: Where are each of the following functions discontinuous?

(a) \( f(x) = \frac{x^2 - x - 2}{x - 2} \) 
(b) \( f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \) 
(c) \( f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \)

Solution:

(a) The function \( f(x) = \frac{x^2 - x - 2}{x - 2} \) is discontinuous at \( x = 2 \), since \( f(x) \) is not defined at this point.

(b) The function \( f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \) is discontinuous at \( x = 0 \), since \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x^2} \) does not exist.

(c) The function \( f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \) is discontinuous at \( x = 2 \), since

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3
\]

which is not equal to \( f(2) = 1 \).

REMARK: Note that the function \( f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases} \) is continuous at \( x = 2 \), since \( \lim_{x \to 2} f(x) = 3 \) which is equal to \( f(2) = 3 \). Moreover, this function is continuous at any \( x \).
DEFINITION 2: A function \( f \) is continuous from the right at a number \( a \) if
\[
\lim_{x \to a^+} f(x) = f(a)
\]
and \( f \) is continuous from the left at \( a \) if
\[
\lim_{x \to a^-} f(x) = f(a)
\]

EXAMPLE: The graph of a function \( g \) is shown.

(a) At which points \( a \) in \( \{0, 1, 2, 3, 4, 5\} \) is \( g \) continuous?
(b) At which points \( a \) in \( \{0, 1, 2, 3, 4, 5\} \) is \( g \) continuous from the right?
(c) At which points \( a \) in \( \{0, 1, 2, 3, 4, 5\} \) is \( g \) continuous from the left?
EXAMPLE: The graph of a function \( g \) is shown.

(a) At which points \( a \) in \( \{0, 1, 2, 3, 4, 5\} \) is \( g \) continuous?

(b) At which points \( a \) in \( \{0, 1, 2, 3, 4, 5\} \) is \( g \) continuous from the right?

(c) At which points \( a \) in \( \{0, 1, 2, 3, 4, 5\} \) is \( g \) continuous from the left?

Solution:

(a) The function \( g \) is continuous at \( a = 0, 2, 5 \). In fact,

(i) The function \( g \) is continuous at \( a = 0 \), since \( \lim_{x \to 0^+} g(x) = g(0) = -1 \).

(ii) The function \( g \) is not continuous at \( a = 1 \), since \( \lim_{x \to 1^-} g(x) \neq \lim_{x \to 1^+} g(x) \).

(iii) The function \( g \) is continuous at \( a = 2 \), since \( \lim_{x \to 2^-} g(x) = \lim_{x \to 2^+} g(x) = g(2) = 0 \).

(iv) The function \( g \) is not continuous at \( a = 3 \), since \( \lim_{x \to 3^-} g(x) \neq \lim_{x \to 3^+} g(x) \).

(v) The function \( g \) is not continuous at \( a = 4 \), since \( g(4) \) does not exist.

(vi) The function \( g \) is continuous at \( a = 5 \), since \( \lim_{x \to 5^-} g(x) = g(5) = 1 \).

(b) The function \( g \) is continuous from the right at \( a = 0, 1, 2, 3 \). In fact,

(i) The function \( g \) is continuous from the right at \( a = 0 \), since \( \lim_{x \to 0^+} g(x) = g(0) \).

(ii) The function \( g \) is continuous from the right at \( a = 1 \), since \( \lim_{x \to 1^+} g(x) = g(1) \).

(iii) The function \( g \) is continuous from the right at \( a = 2 \), since \( \lim_{x \to 2^+} g(x) = g(2) \).

(iv) The function \( g \) is continuous from the right at \( a = 3 \), since \( \lim_{x \to 3^+} g(x) = g(3) \).

(v) The function \( g \) is not continuous from the right at \( a = 4 \), since \( g(4) \) does not exist.

(vi) The function \( g \) is not continuous from the right at \( a = 5 \), since \( \lim_{x \to 5^+} g(x) \) does not exist.

(c) The function \( g \) is continuous from the left at \( a = 2, 5 \). In fact,

(i) The function \( g \) is not continuous from the left at \( a = 0 \), since \( \lim_{x \to 0^-} g(x) \) does not exist.

(ii) The function \( g \) is not continuous from the left at \( a = 1 \), since \( \lim_{x \to 1^-} g(x) \neq g(1) \).

(iii) The function \( g \) is continuous from the left at \( a = 2 \), since \( \lim_{x \to 2^-} g(x) = g(2) \).

(iv) The function \( g \) is not continuous from the left at \( a = 3 \), since \( \lim_{x \to 3^-} g(x) \neq g(3) \).

(v) The function \( g \) is not continuous from the left at \( a = 4 \), since \( g(4) \) does not exist.

(vi) The function \( g \) is continuous from the left at \( a = 5 \), since \( \lim_{x \to 5^-} g(x) = g(5) \).
DEFINITION 3: A function $f$ is **continuous on an interval** if it is continuous at every point in the interval. (If $f$ is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

EXAMPLE: Show that the function $f(x) = \sqrt{16 - x^4}$ is continuous on the interval $[-2, 2]$.

Solution: If $-2 < a < 2$, then using the Limit Laws, we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} \sqrt{16 - x^4} \overset{LL}{=} \sqrt{\lim_{x \to a} (16 - x^4)} \overset{DSP}{=} \sqrt{16 - a^4} = f(a)$$

therefore by Definition 1 the function is continuous. Similarly, since

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \sqrt{16 - x^4} \overset{LL}{=} \sqrt{\lim_{x \to -2^+} (16 - x^4)} \overset{DSP}{=} \sqrt{16 - (-2)^4} = 0 = f(-2)$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \sqrt{16 - x^4} \overset{LL}{=} \sqrt{\lim_{x \to 2^-} (16 - x^4)} \overset{DSP}{=} \sqrt{16 - 2^4} = 0 = f(2)$$

it follows that $f$ is continuous from the right at $-2$ and continuous from the left at 2. Therefore, according to Definition 3, $f$ is continuous on $[-2, 2]$.

THEOREM: If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

- $cf$
- $f \pm g$
- $fg$
- $\frac{f}{g}$ (if $g(a) \neq 0$)

THEOREM:
(a) Any polynomial is continuous everywhere.
(b) Any rational function is continuous wherever it is defined.

In general, the following is true:

THEOREM: The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions.

EXAMPLES:
1. $f(x) = 17x^9 + 5x^2 + x - 22$ is continuous on $(-\infty, \infty)$.
2. $f(x) = \frac{x + 1}{x - 2}$ is continuous on $(-\infty, 2) \cup (2, \infty)$.
3. $f(x) = \frac{x}{x}$ is continuous on $(-\infty, 0) \cup (0, \infty)$.
4. $f(x) = 1$ is continuous on $(-\infty, \infty)$.
5. $f(x) = \frac{7x^5 + x - 2}{x^2 - 4}$ is continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.
6. $f(x) = \sin x + \sqrt{x} - \frac{1}{x - 4}$ is continuous on $[0, 4) \cup (4, \infty)$.
THEOREM: If \( f \) is continuous at \( b \) and \( \lim_{x \to a} g(x) = b \), then \( \lim_{x \to a} f(g(x)) = f(b) \). In other words, \[
\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))
\]

THEOREM: If \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \), then the composite function \( f \circ g \) given by \( (f \circ g)(x) = f(g(x)) \) is continuous at \( a \).

EXAMPLES:

1. The function \( f(x) = \cos(x^2 + 1) \) is continuous on \((−\infty, \infty)\) by the above Theorem, because \( x^2 + 1 \) is continuous on \((−\infty, \infty)\) and \( \cos x \) is continuous on \((−\infty, \infty)\).

2. The function \( f(x) = \sqrt{16 - x^4} \) is continuous on \([-2, 2]\) by the above Theorem, because \( 16 - x^4 \) is continuous on \((−\infty, \infty)\), \( \sqrt{x} \) is continuous on \([0, \infty)\) and \( 16 - x^4 \geq 0 \) on \([-2, 2]\).

THE INTERMEDIATE VALUE THEOREM: Suppose that \( f \) is continuous on the closed interval \([a, b]\) and let \( N \) be any number between \( f(a) \) and \( f(b) \), where \( f(a) \neq f(b) \). Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \).

EXAMPLE: Show that there is a root of the equation \( 3x^7 - 2x^5 + x - 1 = 0 \) between 0 and 1.

Solution: Put \( f(x) = 3x^7 - 2x^5 + x - 1 \). One can check that \( f(0) < 0 \) and \( f(1) > 0 \). From this by the IVT it follows that there exists a number \( c \) in \((0, 1)\) such that \( f(c) = 0 \) since \( f(x) \) is continuous (polynomial) and 0 is between \( f(0) \) and \( f(1) \).

EXAMPLE: Show that there is a root of the equation \( x^9 + x = 5 \).

Solution 1: Put \( f(x) = x^9 + x \). One can check that \( f(1) < 5 \) and \( f(2) > 5 \). From this by the IVT it follows that there exists a number \( c \) in \((1, 2)\) such that \( f(c) = 5 \) since \( f(x) \) is continuous (polynomial) and 5 is between \( f(1) \) and \( f(2) \).

Solution 2: Put \( f(x) = x^9 + x - 5 \). One can check that \( f(1) < 0 \) and \( f(2) > 0 \). From this by the IVT it follows that there exists a number \( c \) in \((1, 2)\) such that \( f(c) = 0 \) since \( f(x) \) is continuous (polynomial) and 0 is between \( f(1) \) and \( f(2) \).