The Limit of a Function

DEFINITION: We write

$$\boxed{\lim_{x\to a} f(x) = L}$$

and say

$$\text{"the limit of } f(x)\text{, as } x \text{ approaches } a\text{, equals } L\text{"}$$

if we can make the values of $f(x)$ arbitrary close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

EXAMPLE:

x approaches 2	x approaches 2	x approaches 2	x approaches 2	x does NOT approach 2	x does NOT approach 2	x approaches 2
1.9	2.1	2.4	2400000000⟵	2.1	2.4	2.4
1.99	2.01	1.98	198000⟵	-2.01⟵	1.98	1.98
1.999	2.001	2.004	200.04⟵	2.001	200.7⟵	200.7⟵
1.9999	2.0001	1.9995	19.995⟵	-2.0001⟵	1.9995	1.9995
1.99999	2.00001	2.00009	2.00009	2.00001	2.00009	2.00009
1.999999	2.000001	1.999999	1.999999	-2.000001⟵	9.87735⟵	9.87735⟵
1.9999999	2.0000001	1.9999997	1.9999997	2.0000001	1.9999997	1.9999997
1.99999999	2.00000001	2.00000004	2.00000004	-2.00000001⟵	20.5736732⟵	2.05736732
1.999999999	2.000000001	2.000000009	2.000000009	2.000000009	2.000000009	2.000000009
1.9999999999	2.0000000001	1.9999999991	1.9999999991	-1.9999999991⟵	1.9999999991	1.9999999991
1.99999999999	2.00000000001	2.00000000002	2.00000000002	2.00000000002	70.76456523⟵	2.00000000002
1.999999999999	2.000000000001	1.999999999994	1.999999999994	-1.999999999994⟵	1.999999999994	1.999999999994

EXAMPLES:

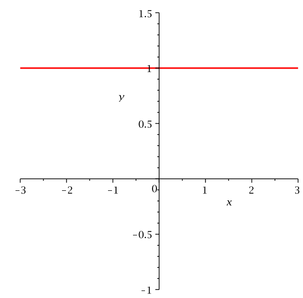
1. Let $f(x) = 1$ and $a = 2$. Estimate the value of $\lim_{x\to a} f(x)$.

Solution: We have

x	$f(x)$
1.9	1
1.99	1
1.999	1
1.9999	1
1.99999	1
1.999999	1
1.9999999	1
1.99999999	1

x	$f(x)$
2.1	1
2.01	1
2.001	1
2.0001	1
2.00001	1
2.000001	1
2.0000001	1
2.00000001	1

x	$f(x)$
2.4	1
1.98	1
2.004	1
1.9995	1
2.00009	1
1.999999	1
1.9999997	1
2.00000004	1



Looking at these tables and at the graph (or simply using common sense) one can conclude that $\lim_{x\to a} f(x) = \lim_{x\to 2} 1 = 1$.

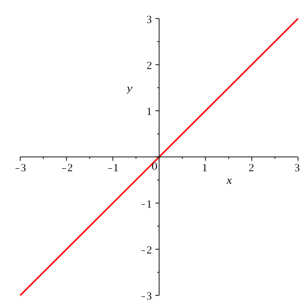
2. Let $f(x) = x$ and $a = 2$. Estimate the value of $\lim_{x\to a} f(x)$.

Solution: We have

x	$f(x)$
1.9	1.9
1.99	1.99
1.999	1.999
1.9999	1.9999
1.99999	1.99999
1.999999	1.999999
1.9999999	1.9999999
1.99999999	1.99999999

x	$f(x)$
2.1	2.1
2.01	2.01
2.001	2.001
2.0001	2.0001
2.00001	2.00001
2.000001	2.000001
2.0000001	2.0000001
2.00000001	2.00000001

x	$f(x)$
2.4	2.4
1.98	1.98
2.004	2.004
1.9995	1.9995
2.00009	2.00009
1.999999	1.999999
1.9999997	1.9999997
2.00000004	2.00000004



Looking at these tables and at the graph (or simply using common sense) one can conclude that $\lim_{x\to a} f(x) = \lim_{x\to 2} x = 2$.

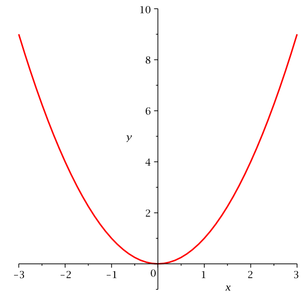
3. Estimate the value of $\lim_{x\to 2} x^2$.

Solution: We have

x	$f(x)$
1.9	3.610000000
1.99	3.960100000
1.999	3.996001000
1.9999	3.999600010
1.99999	3.999960000
1.999999	3.999996000
1.9999999	3.999999600
1.99999999	3.999999960

x	$f(x)$
2.1	4.410000000
2.01	4.040100000
2.001	4.004001000
2.0001	4.000400010
2.00001	4.000040000
2.000001	4.000004000
2.0000001	4.000000400
2.00000001	4.000000040

x	$f(x)$
2.4	4.840000000
1.98	4.161600000
2.004	3.980025000
1.9995	4.001200090
2.00009	3.999800002
1.999999	3.999968000
1.9999997	3.999998800
2.00000004	3.999999720



Looking at these tables and at the graph one can conclude that $\lim_{x\to 2} x^2 = 4$. Note that the value of $f(x) = x^2$ at 2 is also 4.

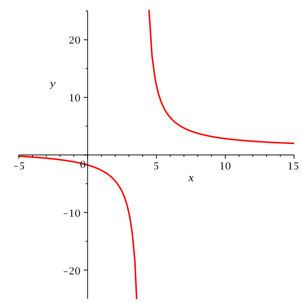
4. Estimate the value of $\lim_{x\to 3} \dfrac{x+7}{x-4}$.

Solution: We have

x	$f(x)$
2.9	-9.000000000
2.99	-9.891089109
2.999	-9.989010989
2.9999	-9.998900110
2.99999	-9.999890001
2.999999	-9.999989000
2.9999999	-9.999998900
2.99999999	-9.999999890

x	$f(x)$
3.1	-11.22222222
3.01	-10.11111111
3.001	-10.01101101
3.0001	-10.00110011
3.00001	-10.00011000
3.000001	-10.00001100
3.0000001	-10.00000110
3.00000001	-10.00000011

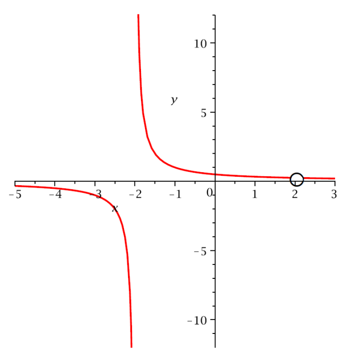
x	$f(x)$
2.9	-5.111111111
3.02	-10.34020619
3.006	-9.956175299
3.0002	-9.996700990
3.00004	-10.00077005
2.999998	-9.999978000
2.9999994	-9.999990100
2.99999996	-9.999999560



Looking at these tables and at the graph one can conclude that $\lim_{x\to 3} \dfrac{x+7}{x-4} = -10$. Note that the value of $f(x) = \dfrac{x+7}{x-4}$ at 3 is also -10.

5. Estimate the value of $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$.

Solution: Note that $f(x) = \dfrac{x-2}{x^2-4}$ does not exist at 2. We have



x	$f(x)$
1.9	0.2564102564
1.99	0.2506265664
1.999	0.2500625156
1.9999	0.2500062502
1.99999	0.2500000000
1.999999	0.2500000000
1.9999999	0.2500000000
1.99999999	0.2500000000

x	$f(x)$
2.1	0.2439024390
2.01	0.2493765586
2.001	0.2499375156
2.0001	0.2499937502
2.00001	0.2500000000
2.000001	0.2500000000
2.0000001	0.2500000000
2.00000001	0.2500000000

x	$f(x)$
1.2	0.2564102564
1.93	0.2469135802
2.008	0.2503128911
2.0007	0.2499687539
2.00009	0.2500006250
2.000009	0.2499995000
2.0000006	0.2499999500
1.999999970	0.2499999994

Looking at these tables and at the graph one can conclude that $\lim_{x\to 2} \dfrac{x-2}{x^2-4} = 0.25$.

REMARK: Note that if $x \neq \pm 2$, then

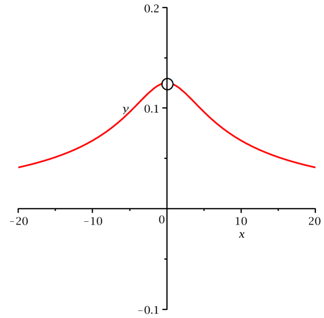
$$\frac{x-2}{x^2-4} = \frac{x-2}{x^2-2^2} = \frac{(x-2)\cdot 1}{(x-2)(x+2)} = \frac{1}{x+2}$$

therefore $\lim_{x\to 2} \dfrac{x-2}{x^2-4} = \lim_{x\to 2} \dfrac{1}{x+2}$, which explains the above answer.

6. Estimate the value of $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+16}-4}{x^2}$.

Solution: Note that $f(x)=\dfrac{\sqrt{x^2+16}-4}{x^2}$ does not exist at 0. We have

x	$f(x)$
±0.1	0.1249805
±0.01	0.1250000
±0.001	0.1250000
±0.0001	0.1250000
±0.00001	0.1250000
±0.000001	0.1250000
±0.0000001	0.1250000
±0.00000001	0.1250000



Looking at this table and the graph one can conclude that $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+16}-4}{x^2}=0.125$.

REMARK: Note that if $x\neq 0$, then

$$\frac{\sqrt{x^2+16}-4}{x^2}=\frac{\sqrt{x^2+16}-4}{x^2+16-16}=\frac{\sqrt{x^2+16}-4}{(\sqrt{x^2+16})^2-4^2}$$
$$=\frac{\sqrt{x^2+16}-4}{(\sqrt{x^2+16}-4)(\sqrt{x^2+16}+4)}=\frac{1}{\sqrt{x^2+16}+4}$$

or

$$\frac{\sqrt{x^2+16}-4}{x^2}=\frac{(\sqrt{x^2+16}-4)(\sqrt{x^2+16}+4)}{x^2(\sqrt{x^2+16}+4)}$$
$$=\frac{x^2+16-4^2}{x^2(\sqrt{x^2+16}+4)}=\frac{x^2}{x^2(\sqrt{x^2+16}+4)}=\frac{1}{\sqrt{x^2+16}+4}$$

therefore

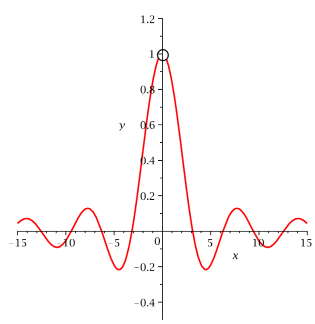
$$\lim_{x\to 0}\frac{\sqrt{x^2+16}-4}{x^2}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+16}+4}$$

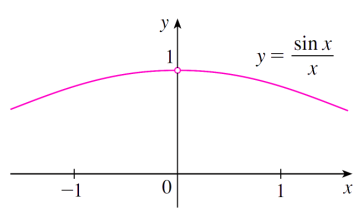
which explains the above answer.

7. Estimate the value of $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Solution: Note that $f(x)=\dfrac{\sin x}{x}$ does not exist at 0. We have

x	$f(x)$
±0.1	0.998334166
±0.01	0.999983333
±0.001	0.999999833
±0.0001	0.999999833
±0.00001	1.000000000
±0.000001	1.000000000
±0.0000001	1.000000000
±0.00000001	1.000000000





Looking at this table and the graph one can conclude that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$.

8. Investigate $\lim\limits_{x\to 0} \sin \dfrac{\pi}{x}$.

Solution: Note that $f(x) = \sin \dfrac{\pi}{x}$ does not exist at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin \frac{\pi}{1} = \sin \pi = 0 \qquad f\left(\frac{1}{2}\right) = \sin \frac{\pi}{1/2} = \sin 2\pi = 0$$

$$f\left(\frac{1}{3}\right) = \sin \frac{\pi}{1/3} = \sin 3\pi = 0 \qquad f\left(\frac{1}{4}\right) = \sin \frac{\pi}{1/4} = \sin 4\pi = 0$$

$$f(0.1) = \sin \frac{\pi}{0.1} = \sin 10\pi = 0 \qquad f(0.01) = \sin \frac{\pi}{0.01} = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that

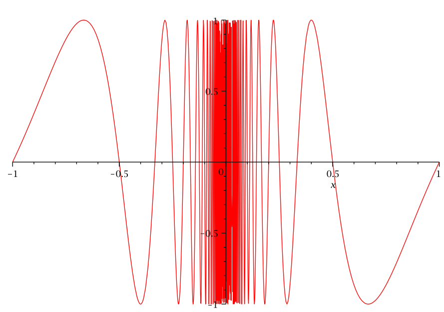
$$\lim_{x\to 0} \sin \frac{\pi}{x} = 0$$

but this guess is wrong. Note that although

$$f\left(\frac{1}{n}\right) = \sin \frac{\pi}{1/n} = \sin n\pi = 0 \quad \text{for any integer } n$$

it is also true that $f(x) \neq 0$ for infinitely many values of x that approach 0. You can see this from the tables and the graph below.

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.1	0	0.2	0	0.3	-0.8660254028	0.7	-0.9749279126
0.01	0	0.02	0	0.03	-0.8660253940	0.07	0.7818314941
0.001	0	0.002	0	0.003	-0.8660253055	0.007	0.4338835705
0.0001	0	0.0002	0	0.0003	-0.8660244208	0.0007	0.9749274957
0.00001	0	0.00002	0	0.00003	-0.8660155737	0.00007	-0.7818198123
0.000001	0	0.000002	0	0.000003	-0.8659270882	0.000007	-0.4340523680
0.0000001	0	0.0000002	0	0.0000003	-0.8659270882	0.0000007	-0.9753427019
0.00000001	0	0.00000002	0	0.00000003	-0.8560287927	0.00000007	0.7933638615

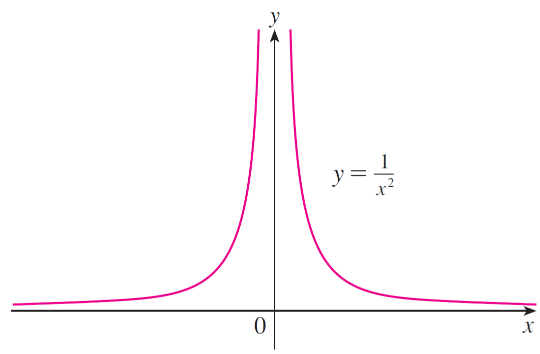


Looking at these tables and the graph one can conclude that $\lim\limits_{x\to 0} \sin \dfrac{\pi}{x}$ does not exist.

9. Investigate $\lim\limits_{x\to 0}\dfrac{1}{x^2}$.

Solution: Note that $f(x)=\dfrac{1}{x^2}$ does not exist at 0. We have

x	$f(x)$
$\pm$ 0.1	100
$\pm$ 0.01	10000
$\pm$ 0.001	1000000
$\pm$ 0.0001	100000000
$\pm$ 0.00001	10000000000
$\pm$ 0.000001	1000000000000
$\pm$ 0.0000001	100000000000000
$\pm$ 0.00000001	10000000000000000



Looking at this table and the graph one can conclude that this limit does not exist.

EXAMPLES:

1. $\lim\limits_{x\to -2} x^3$

2. $\lim\limits_{x\to 1}\dfrac{1}{1+x}$

3. $\lim\limits_{x\to -1}\dfrac{1}{1+x}$

4. $\lim\limits_{x\to 0}\sin x$

5. $\lim\limits_{x\to \pi/2}\sin x$

6. $\lim\limits_{x\to \pi/2}\tan x$

7. $\lim\limits_{x\to 0}\cos\left(\dfrac{\pi}{x}\right)$

8. $\lim\limits_{x\to -1}|x|$

9. $\lim\limits_{x\to 1}\dfrac{1}{x}$

10. $\lim\limits_{x\to 0}\dfrac{1}{x}$

EXAMPLES:

1. $\lim_{x\to -2} x^3 = -8$

2. $\lim_{x\to 1} \frac{1}{1+x} = \frac{1}{2}$

3. $\lim_{x\to -1} \frac{1}{1+x}$ D.N.E.

4. $\lim_{x\to 0} \sin x = 0$

5. $\lim_{x\to \pi/2} \sin x = 1$

6. $\lim_{x\to \pi/2} \tan x$ D.N.E.

7. $\lim_{x\to 0} \cos\left(\frac{\pi}{x}\right)$ D.N.E.

8. $\lim_{x\to -1} |x| = 1$

9. $\lim_{x\to 1} \frac{1}{x} = 1$

10. $\lim_{x\to 0} \frac{1}{x}$ D.N.E.

One-sided Limits

DEFINITION: We write

$$\boxed{\lim_{x\to a^-} f(x) = L}$$

and say the **left-hand limit of $f(x)$ as x approaches a** is equal to L if we can make the values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and x less than a.

Similarly, we write

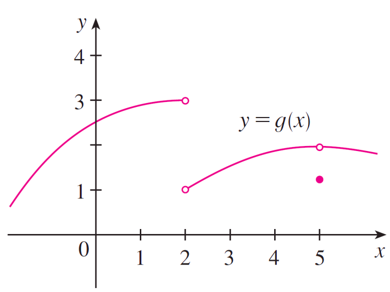
$$\boxed{\lim_{x\to a^+} f(x) = L}$$

and say the **right-hand limit of $f(x)$ as x approaches a** is equal to L if we can make the values of $f(x)$ arbitrary close to L by taking x to be sufficiently close to a and x greater than a.

IMPORTANT:

$$\boxed{\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L}$$

EXAMPLE: The graph of a function g is shown below. Use it to state the values (if they exist) of the following:



(a) $\lim_{x\to 2^-} g(x)$

(b) $\lim_{x\to 2^+} g(x)$

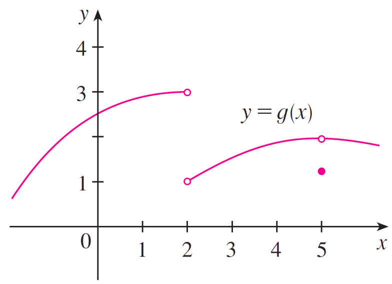
(c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$

(e) $\lim_{x\to 5^+} g(x)$

(f) $\lim_{x\to 5} g(x)$

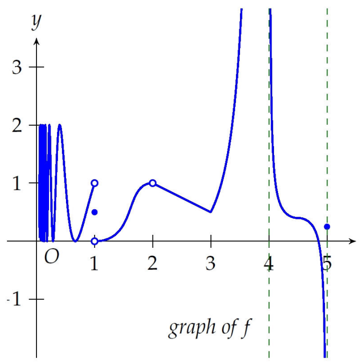
EXAMPLE: The graph of a function g is shown below. Use it to state the values (if they exist) of the following:



(a) $\lim\limits_{x\to 2^-} g(x) = 3$ (b) $\lim\limits_{x\to 2^+} g(x) = 1$ (c) $\lim\limits_{x\to 2} g(x)$ D.N.E.

(d) $\lim\limits_{x\to 5^-} g(x) = 2$ (e) $\lim\limits_{x\to 5^+} g(x) = 2$ (f) $\lim\limits_{x\to 5} g(x) = 2$

EXAMPLE: The graph of a function f is shown below. Use it to state the values (if they exist) of the following:



(a) $\lim\limits_{x\to 0^-} f(x)$ (b) $\lim\limits_{x\to 0^+} f(x)$ (c) $\lim\limits_{x\to 0} f(x)$ (d) $f(0)$

(e) $\lim\limits_{x\to 1^-} f(x)$ (f) $\lim\limits_{x\to 1^+} f(x)$ (g) $\lim\limits_{x\to 1} f(x)$ (h) $f(1)$

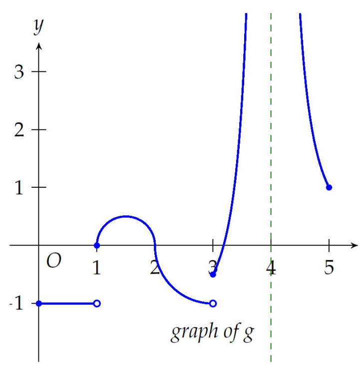
(i) $\lim\limits_{x\to 2^-} f(x)$ (j) $\lim\limits_{x\to 2^+} f(x)$ (k) $\lim\limits_{x\to 2} f(x)$ (l) $f(2)$

(m) $\lim\limits_{x\to 3^-} f(x)$ (n) $\lim\limits_{x\to 3^+} f(x)$ (o) $\lim\limits_{x\to 3} f(x)$ (p) $f(3)$

(q) $\lim\limits_{x\to 4^-} f(x)$ (r) $\lim\limits_{x\to 4^+} f(x)$ (s) $\lim\limits_{x\to 4} f(x)$ (t) $f(4)$

(u) $\lim\limits_{x\to 5^-} f(x)$ (v) $\lim\limits_{x\to 5^+} f(x)$ (w) $\lim\limits_{x\to 5} f(x)$ (x) $f(5)$

EXAMPLE: The graph of a function g is shown below. Use it to state the values (if they exist) of the following:



(a) $\lim\limits_{x\to 0^-} g(x)$ (b) $\lim\limits_{x\to 0^+} g(x)$ (c) $\lim\limits_{x\to 0} g(x)$ (d) $g(0)$

(e) $\lim\limits_{x\to 1^-} g(x)$ (f) $\lim\limits_{x\to 1^+} g(x)$ (g) $\lim\limits_{x\to 1} g(x)$ (h) $g(1)$

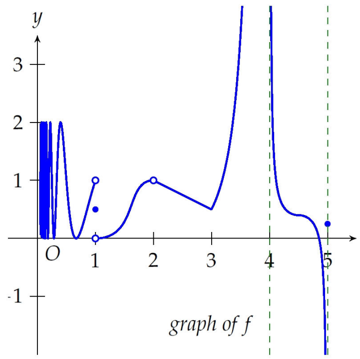
(i) $\lim\limits_{x\to 2^-} g(x)$ (j) $\lim\limits_{x\to 2^+} g(x)$ (k) $\lim\limits_{x\to 2} g(x)$ (l) $g(2)$

(m) $\lim\limits_{x\to 3^-} g(x)$ (n) $\lim\limits_{x\to 3^+} g(x)$ (o) $\lim\limits_{x\to 3} g(x)$ (p) $g(3)$

(q) $\lim\limits_{x\to 4^-} g(x)$ (r) $\lim\limits_{x\to 4^+} g(x)$ (s) $\lim\limits_{x\to 4} g(x)$ (t) $g(4)$

(u) $\lim\limits_{x\to 5^-} g(x)$ (v) $\lim\limits_{x\to 5^+} g(x)$ (w) $\lim\limits_{x\to 5} g(x)$ (x) $g(5)$

EXAMPLE: The graph of a function f is shown below. Use it to state the values (if they exist) of the following:



(a) $\lim_{x\to 0^-} f(x)$ D.N.E. (b) $\lim_{x\to 0^+} f(x)$ D.N.E. (c) $\lim_{x\to 0} f(x)$ D.N.E. (d) $f(0)$ D.N.E.

(e) $\lim_{x\to 1^-} f(x) = 1$ (f) $\lim_{x\to 1^+} f(x) = 0$ (g) $\lim_{x\to 1} f(x)$ D.N.E. (h) $f(1) = 0.5$

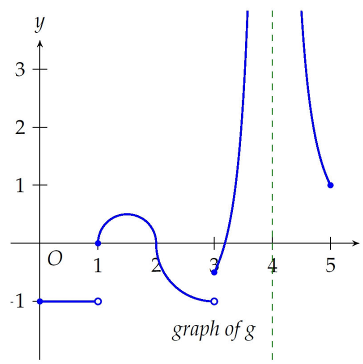
(i) $\lim_{x\to 2^-} f(x) = 1$ (j) $\lim_{x\to 2^+} f(x) = 1$ (k) $\lim_{x\to 2} f(x) = 1$ (l) $f(2)$ D.N.E.

(m) $\lim_{x\to 3^-} f(x) = 0.5$ (n) $\lim_{x\to 3^+} f(x) = 0.5$ (o) $\lim_{x\to 3} f(x) = 0.5$ (p) $f(3) = 0.5$

(q) $\lim_{x\to 4^-} f(x)$ D.N.E. (r) $\lim_{x\to 4^+} f(x)$ D.N.E. (s) $\lim_{x\to 4} f(x)$ D.N.E. (t) $f(4)$ D.N.E.

(u) $\lim_{x\to 5^-} f(x)$ D.N.E. (v) $\lim_{x\to 5^+} f(x)$ D.N.E. (w) $\lim_{x\to 5} f(x)$ D.N.E. (x) $f(5) = 0.25$

EXAMPLE: The graph of a function g is shown below. Use it to state the values (if they exist) of the following:



(a) $\lim_{x\to 0^-} g(x)$ D.N.E. (b) $\lim_{x\to 0^+} g(x) = -1$ (c) $\lim_{x\to 0} g(x)$ D.N.E. (d) $g(0) = -1$

(e) $\lim_{x\to 1^-} g(x) = -1$ (f) $\lim_{x\to 1^+} g(x) = 0$ (g) $\lim_{x\to 1} g(x)$ D.N.E. (h) $g(1) = 0$

(i) $\lim_{x\to 2^-} g(x) = 0$ (j) $\lim_{x\to 2^+} g(x) = 0$ (k) $\lim_{x\to 2} g(x) = 0$ (l) $g(2) = 0$

(m) $\lim_{x\to 3^-} g(x) = -1$ (n) $\lim_{x\to 3^+} g(x) = -0.5$ (o) $\lim_{x\to 3} g(x)$ D.N.E. (p) $g(3) = -0.5$

(q) $\lim_{x\to 4^-} g(x)$ D.N.E. (r) $\lim_{x\to 4^+} g(x)$ D.N.E. (s) $\lim_{x\to 4} g(x)$ D.N.E. (t) $g(4)$ D.N.E.

(u) $\lim_{x\to 5^-} g(x) = 1$ (v) $\lim_{x\to 5^+} g(x)$ D.N.E. (w) $\lim_{x\to 5} g(x)$ D.N.E. (x) $g(5) = 1$