

A Catalog of Essential Functions

In this course we consider 6 groups of important functions:

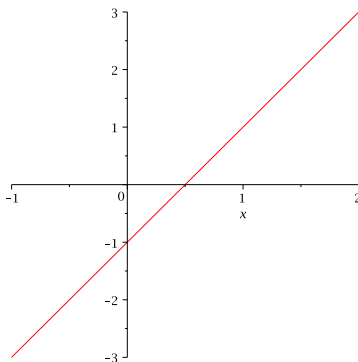
1. Linear Functions
2. Polynomials
3. Power functions
4. Rational functions
5. Trigonometric functions
6. Exponential/Logarithmic functions

EXAMPLES:

1. Linear Functions

$$f(x) = mx + b$$

where m is the slope and b is the y -intercept. Its graph is a straight line:

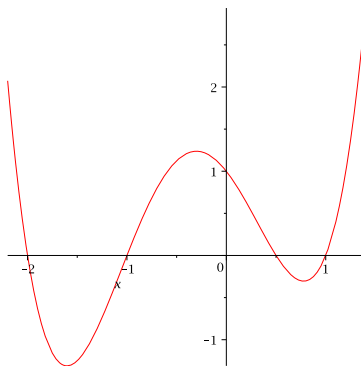


2. Polynomials:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

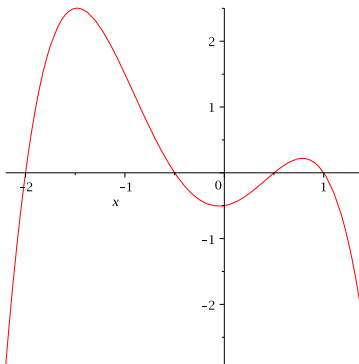
where $a_n, a_{n-1}, \dots, a_2, a_1, a_0$ are constants called the coefficients of $P(x)$ and n is the degree of $P(x)$ (if $a_n \neq 0$).

(a) If $a_n > 0$ and n is even, then its graph is



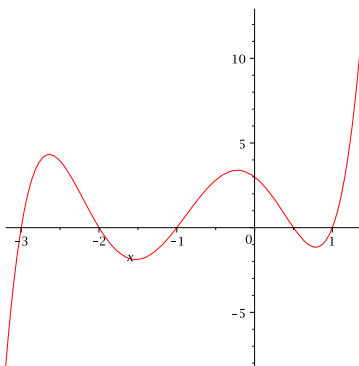
For example, here is a graph of $P(x) = x^4 + \frac{3}{2}x^3 - 2x^2 - \frac{3}{2}x + 1$ with $a_4 = 1 > 0$ and even degree= 4.

(b) If $a_n < 0$ and n is even, then its graph is



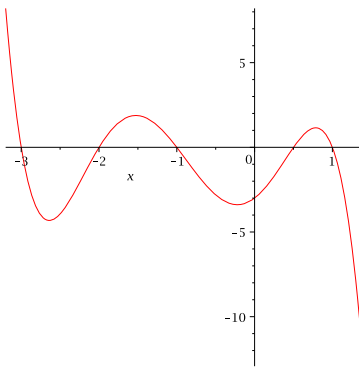
For example, here is a graph of $P(x) = -x^4 - x^3 + \frac{9}{4}x^2 + \frac{1}{4}x - \frac{1}{2}$ with $a_4 = -1 < 0$ and even degree= 4.

(c) If $a_n > 0$ and n is odd, then its graph is



For example, here is a graph of $P(x) = x^5 + \frac{9}{2}x^4 + \frac{5}{2}x^3 - \frac{15}{2}x^2 - \frac{7}{2}x + 3$ with $a_5 = 1 > 0$ and odd degree= 5.

(d) If $a_n < 0$ and n is odd, then its graph is



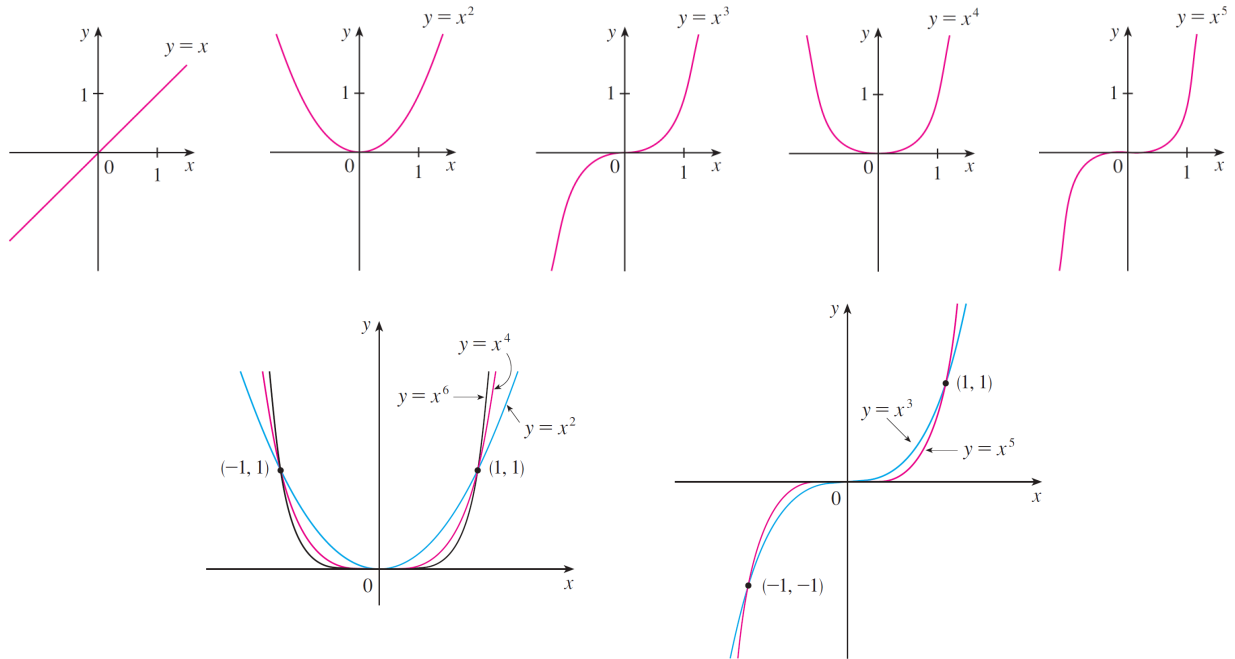
For example, here is a graph of $P(x) = -x^5 - \frac{9}{2}x^4 - \frac{5}{2}x^3 + \frac{15}{2}x^2 + \frac{7}{2}x - 3$ with $a_5 = -1 < 0$ and odd degree= 5.

3. Power functions:

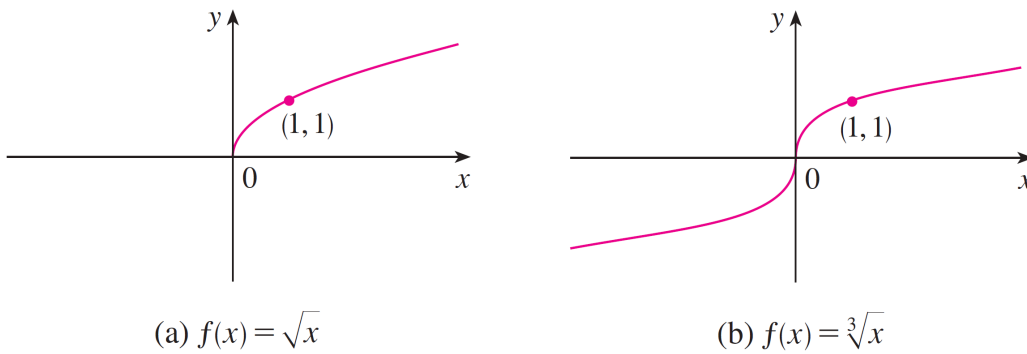
$$f(x) = x^a$$

where a is a constant. Here we distinguish three main cases:

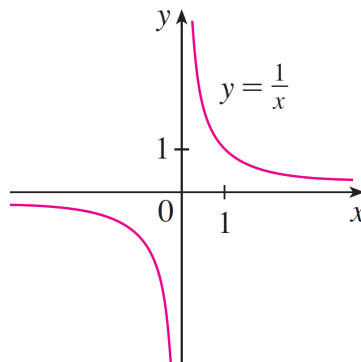
(i) $a = n$, where n is a positive integer



(ii) $a = 1/n$, where n is a positive integer



(iii) $a = -1$

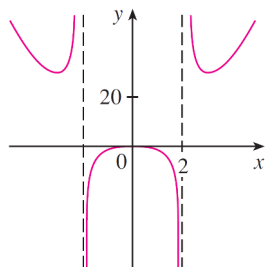


4. Rational functions:

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x), Q(x)$ are polynomials.

EXAMPLES: $f(x) = \frac{1}{x}$, $g(x) = \frac{x+1}{x-3}$, $h(x) = \frac{3x^2 - 5x + 1}{x^3 + 1}$, etc.

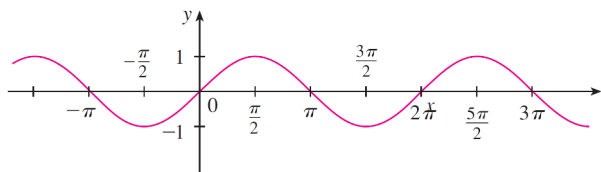


$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

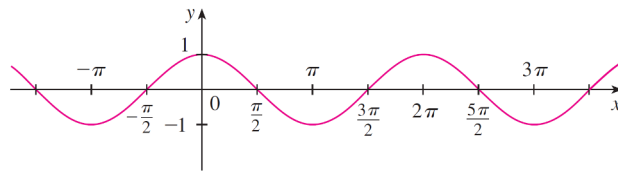
5. Trigonometric functions:

In this course it is important to know graphs and basic properties of the following trigonometric functions:

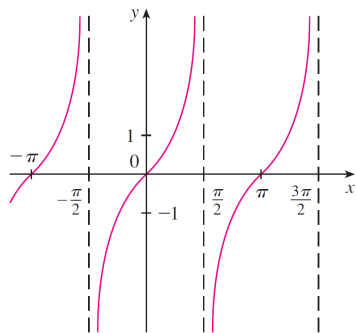
$$\sin x, \quad \cos x, \quad \tan x, \quad \cot x, \quad \sec x, \quad \csc x$$



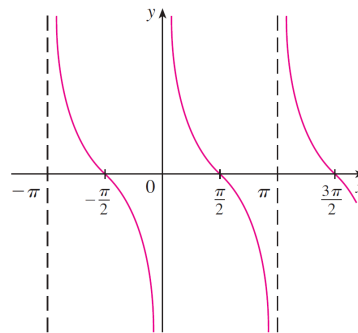
(a) $f(x) = \sin x$



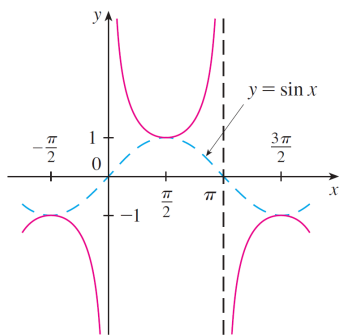
(b) $g(x) = \cos x$



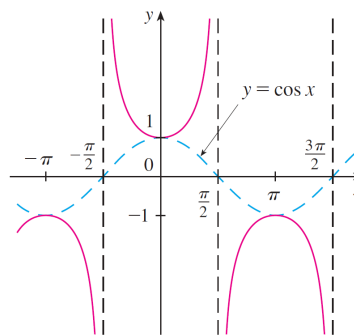
(a) $y = \tan x$



(b) $y = \cot x$



(c) $y = \csc x$

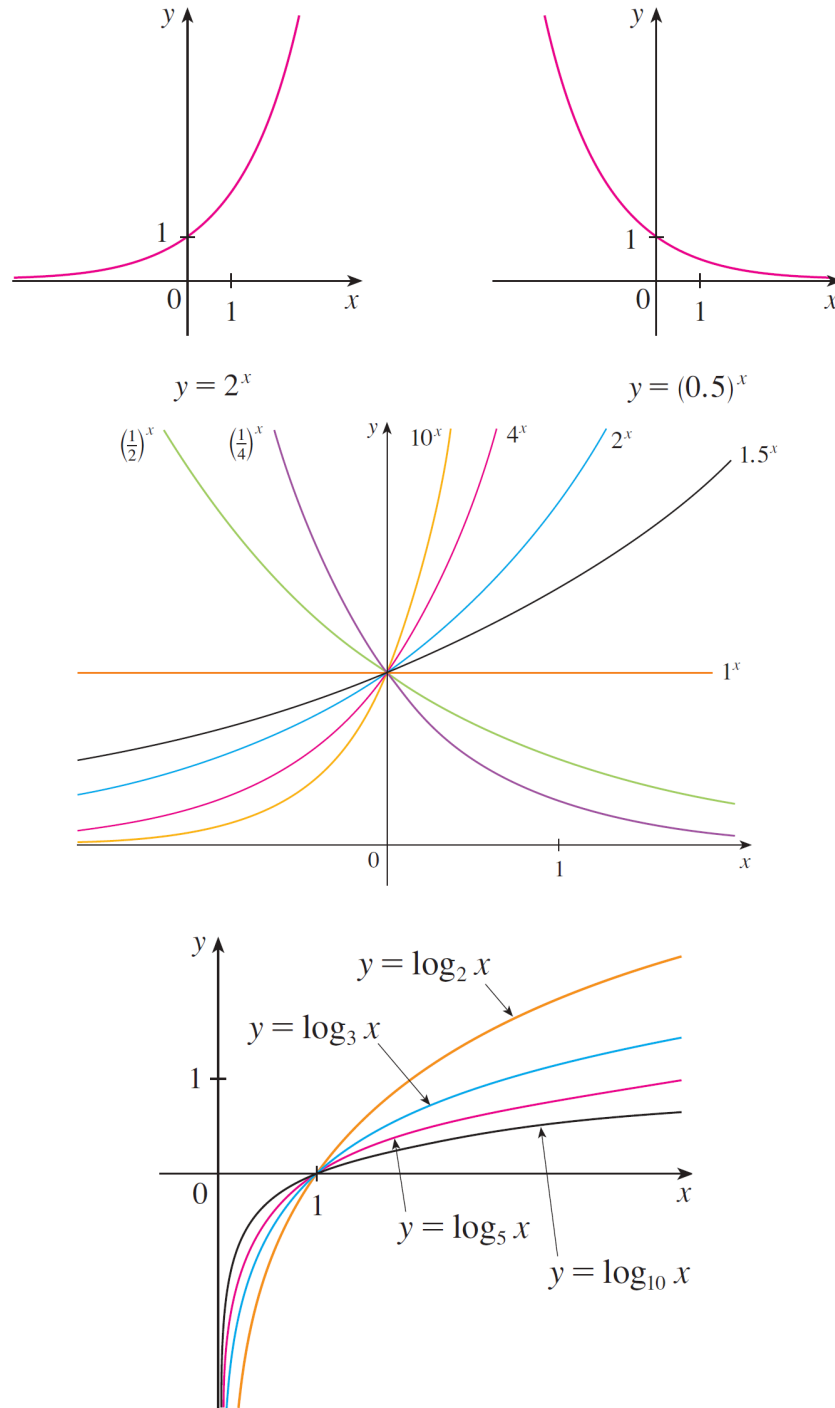


(d) $y = \sec x$

6. Exponential and Logarithmic functions:

$$f(x) = a^x, \quad f(x) = \log_a x$$

where a is a positive constant.



IMPORTANT: Do NOT confuse power functions and exponential functions!

Transformations of Functions

Vertical and Horizontal Shifts: Suppose $c > 0$. To obtain the graph of

- $y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
- $y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
- $y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
- $y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

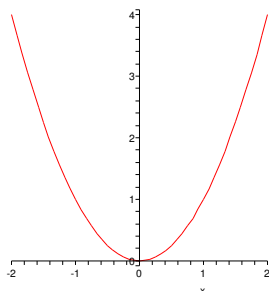
Vertical and Horizontal Stretching and Reflecting: Suppose $c > 0$. To obtain the graph of

- $y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
- $y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
- $y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
- $y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
- $y = -f(x)$, reflect the graph of $y = f(x)$ about the x -axis
- $y = f(-x)$, reflect the graph of $y = f(x)$ about the y -axis

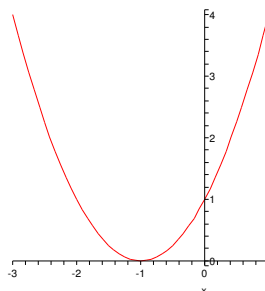
EXAMPLES:

1. Given the graph of $f(x) = x^2$, use transformations to graph $f(x) = (x + 1)^2$.

Step 1: $f(x) = x^2$

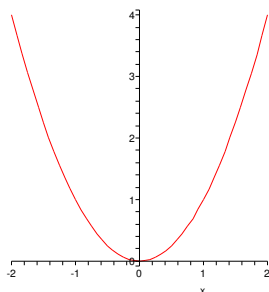


Step 2: $f(x) = (x + 1)^2$ (horizontal shift)

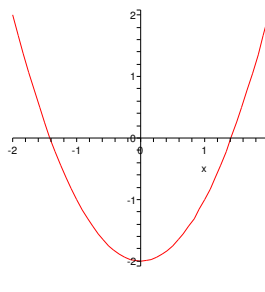


2. Given the graph of $f(x) = x^2$, use transformations to graph $f(x) = x^2 - 2$.

Step 1: $f(x) = x^2$

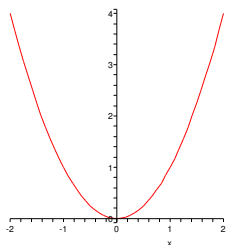


Step 2: $f(x) = x^2 - 2$ (vertical shift)

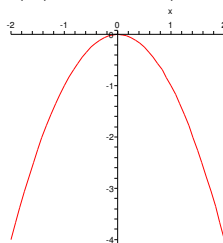


3. Given the graph of $f(x) = x^2$, use transformations to graph $f(x) = -x^2$.

Step 1: $f(x) = x^2$

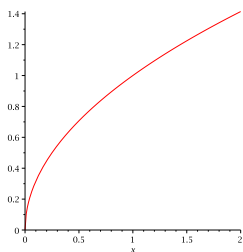


Step 2: $f(x) = -x^2$ (reflection about the x -axis)

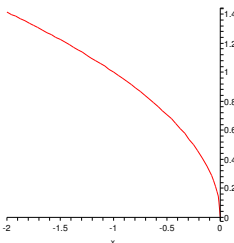


4. Given the graph of $f(x) = \sqrt{x}$, use transformations to graph $f(x) = \sqrt{-x}$.

Step 1: $f(x) = \sqrt{x}$

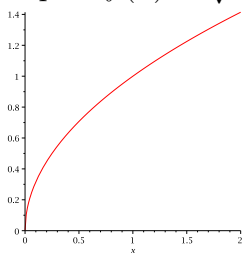


Step 2: $f(x) = \sqrt{-x}$ (reflection about the y -axis)

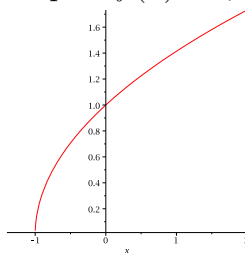


5. Given the graph of $f(x) = \sqrt{x}$, use transformations to graph $f(x) = 1 - \sqrt{1+x}$.

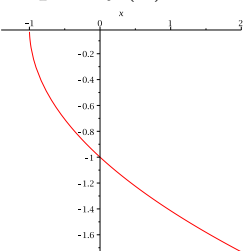
Step 1: $f(x) = \sqrt{x}$



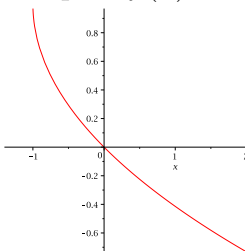
Step 2: $f(x) = \sqrt{1+x}$ (horizontal shift)



Step 3: $f(x) = -\sqrt{1+x}$ (reflection)



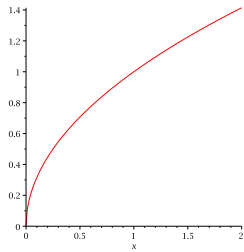
Step 4: $f(x) = 1 - \sqrt{1+x}$ (vertical shift)



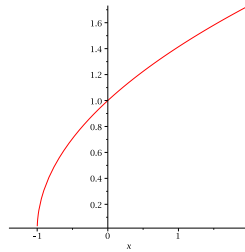
6. Given the graph of $f(x) = \sqrt{x}$, use transformations to graph $f(x) = 1 - \sqrt{1-x}$.

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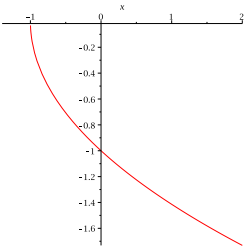
Step 1: $f(x) = \sqrt{x}$



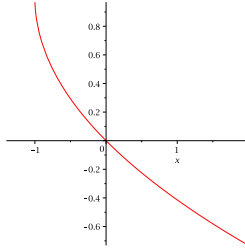
Step 2: $f(x) = \sqrt{1+x}$ (horizontal shift)



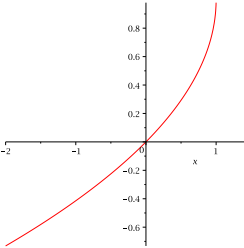
Step 3: $f(x) = -\sqrt{1+x}$ (reflection)



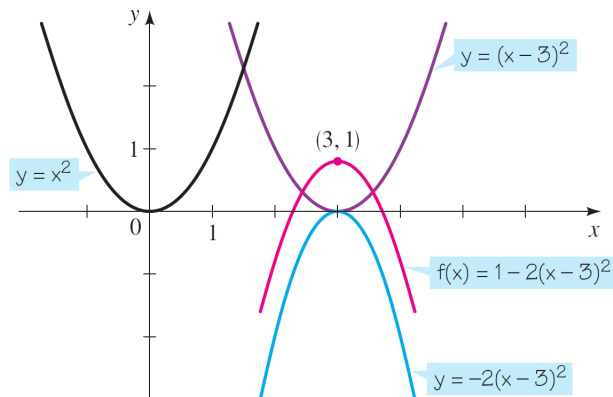
Step 4: $f(x) = 1 - \sqrt{1+x}$ (vertical shift)



Step 5: $f(x) = 1 - \sqrt{1-x}$ (reflection about the y -axis)



7. Sketch the graph of the function $f(x) = 1 - 2(x - 3)^2$.



Combinations of functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Algebra of Functions

Let f and g be functions with domains A and B . Then the functions $f + g$, $f - g$, fg , and f/g are defined as follows.

$$(f + g)(x) = f(x) + g(x) \quad \text{Domain } A \cap B$$

$$(f - g)(x) = f(x) - g(x) \quad \text{Domain } A \cap B$$

$$(fg)(x) = f(x)g(x) \quad \text{Domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \text{Domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

EXAMPLE: The domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$, the domain of $g(x) = \sqrt{1-x}$ is $B = (-\infty, 1]$, and the domain of $h(x) = \sqrt{x-1}$ is $C = [1, \infty)$, so the domain of

$$(f - g)(x) = \sqrt{x} - \sqrt{1-x} \quad \text{is } A \cap B = [0, 1]$$

and

$$(f - h)(x) = \sqrt{x} - \sqrt{x-1} \quad \text{is } A \cap C = [1, \infty)$$

EXAMPLE: If $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function

$$(f/g)(x) = x^2/(x-1) \quad \text{is } \{x \mid x \neq 1\} \text{ or } (-\infty, 1) \cup (1, \infty)$$

There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x . We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g .

Composition of Functions

Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

EXAMPLE: If $f(x) = x^2 + 1$ and $g(x) = x - 3$, find the following.

- (a) $f \circ f$ (b) $f \circ g$ (c) $g \circ f$ (d) $g \circ g$ (e) $f(g(2))$ (f) $g(f(2))$

EXAMPLE: If $f(x) = x^2 + 1$ and $g(x) = x - 3$, find the following.

$$(a) f \circ f \quad (b) f \circ g \quad (c) g \circ f \quad (d) g \circ g \quad (e) f(g(2)) \quad (f) g(f(2))$$

Solution: We have

$$(a) f \circ f = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2$$

$$(b) f \circ g = (x - 3)^2 + 1 = x^2 - 6x + 10$$

$$(c) g \circ f = x^2 + 1 - 3 = x^2 - 2$$

$$(d) g \circ g = x - 3 - 3 = x - 6$$

$$(e) f(g(2)) = (2 - 3)^2 + 1 = 2$$

$$(f) g(f(2)) = 2^2 - 2 = 2$$

EXAMPLE: If $f(x) = x$ and $g(x) = 1$, then

$$f \circ f = x \quad f \circ g = 1 \quad g \circ f = 1 \quad g \circ g = 1$$

REMARK: You can see from the Examples above that sometimes $f \circ g = g \circ f$, but, in general, $f \circ g \neq g \circ f$.

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

EXAMPLE: If $f(x) = x^2$ and $g(x) = \sqrt{x}$, then

$$f \circ f = (x^2)^2 = x^4 \quad f \circ g = x, \quad x \geq 0 \quad g \circ f = |x| \quad g \circ g = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

(of course, the domain of $g \circ g = \sqrt[4]{x}$ is all nonnegative numbers).

EXAMPLE: If $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$, then

$$f \circ f = (x^3)^3 = x^9 \quad f \circ g = x \quad g \circ f = x \quad g \circ g = \sqrt[3]{\sqrt[3]{x}} = \sqrt[9]{x}$$

EXAMPLE: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, find each function and its domain.

$$(a) f \circ g \quad (b) g \circ f \quad (c) f \circ f \quad (d) g \circ g$$

EXAMPLE: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each function and its domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

Solution:

(a) We have

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = \sqrt[4]{2-x}$$

The domain of $f \circ g$ is $\{x \mid 2-x \geq 0\} = \{x \mid x \leq 2\} = (-\infty, 2]$.

(b) We have

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{2-\sqrt{x}}$$

For \sqrt{x} to be defined we must have $x \geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2-\sqrt{x} \geq 0$, that is, $\sqrt{x} \leq 2$, or $x \leq 4$. Thus we have $0 \leq x \leq 4$, so the domain of $g \circ f$ is the closed interval $[0, 4]$.

(c) We have

$$(f \circ f)(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

The domain of $f \circ f$ is $[0, \infty)$.

(d) We have

$$(g \circ g)(x) = g(g(x)) = g(\sqrt{2-x}) = \sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2-x \geq 0$ and $2-\sqrt{2-x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to $\sqrt{2-x} \leq 2$, or $2-x \leq 4$, or $x \geq -2$. Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

EXAMPLE: Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x+3$.

Solution: We have

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^{10}) = \frac{(x+3)^{10}}{(x+3)^{10}+1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE: Given $F(x) = \frac{4}{(x+9)^2}$, find functions f, g , and h such that $F = f \circ g \circ h$.

EXAMPLE: Given $F(x) = \frac{4}{(x+9)^2}$, find functions f, g , and h such that $F = f \circ g \circ h$.

Solution 1: The formula for F says: First add 9, then square $x + 9$, and finally divide 4 by the result. So we let

$$f(x) = \frac{4}{x}, \quad g(x) = x^2, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f((x + 9)^2) = \frac{4}{(x + 9)^2} = F(x)$$

Solution 2: Here is an other way to look at F : First add 9, then divide 2 by $x + 9$, and finally square the result. So we let

$$f(x) = x^2, \quad g(x) = \frac{2}{x}, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f\left(\frac{2}{x + 9}\right) = \left(\frac{2}{x + 9}\right)^2 = \frac{4}{(x + 9)^2} = F(x)$$