A Catalog of Essential Functions

In this course we consider 6 groups of important functions:

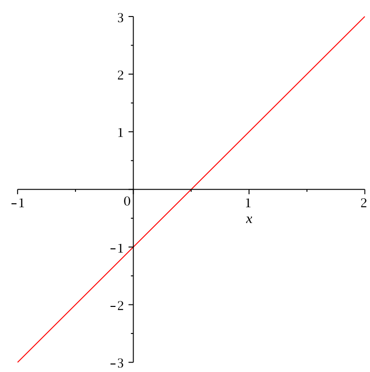
1. Linear Functions
2. Polynomials
3. Power functions
4. Rational functions
5. Trigonometric functions
6. Exponential/Logarithmic functions

EXAMPLES:

1. Linear Functions

$$\boxed{f(x) = mx + b}$$

where m is the slope and b is the y-intercept. Its graph is a straight line:

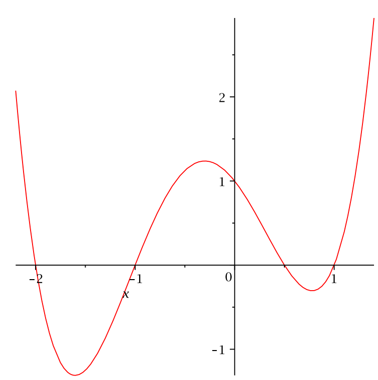


2. Polynomials:

$$\boxed{P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0}$$

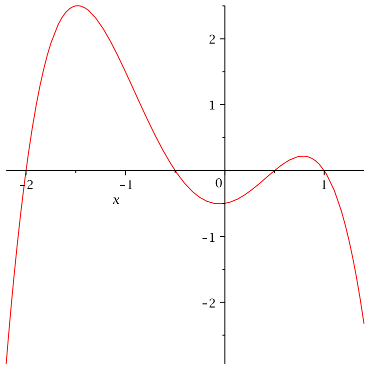
where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants called the coefficients of $P(x)$ and n is the degree of $P(x)$ (if $a_n \neq 0$).

(a) If $a_n > 0$ and n is even, then its graph is



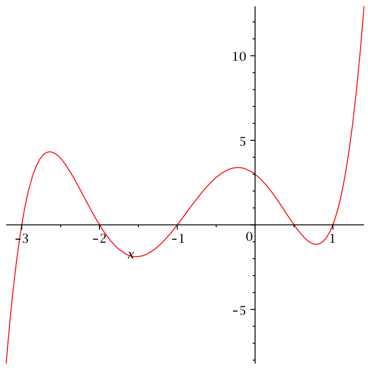
For example, here is a graph of $P(x) = x^4 + \dfrac{3}{2} x^3 - 2 x^2 - \dfrac{3}{2} x + 1$ with $a_4 = 1 > 0$ and even degree$= 4$.

(b) If $a_n < 0$ and n is even, then its graph is



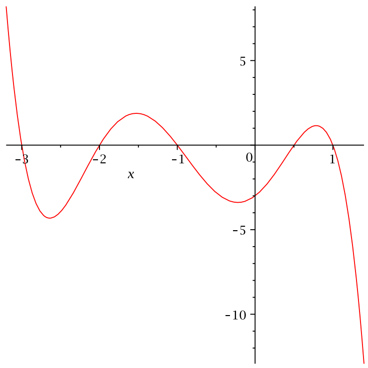
For example, here is a graph of $P(x) = -x^4 - x^3 + \dfrac{9}{4}x^2 + \dfrac{1}{4}x - \dfrac{1}{2}$ with $a_4 = -1 < 0$ and even degree$= 4$.

(c) If $a_n > 0$ and n is odd, then its graph is



For example, here is a graph of $P(x) = x^5 + \dfrac{9}{2}x^4 + \dfrac{5}{2}x^3 - \dfrac{15}{2}x^2 - \dfrac{7}{2}x + 3$ with $a_5 = 1 > 0$ and odd degree$= 5$.

(d) If $a_n < 0$ and n is odd, then its graph is



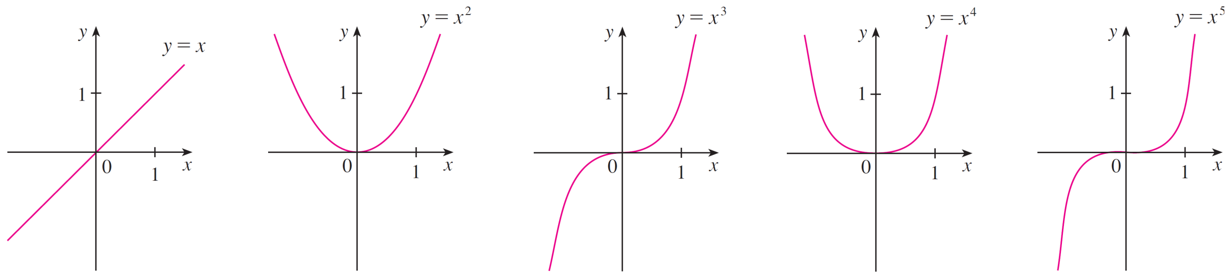
For example, here is a graph of $P(x) = -x^5 - \dfrac{9}{2}x^4 - \dfrac{5}{2}x^3 + \dfrac{15}{2}x^2 + \dfrac{7}{2}x - 3$ with $a_5 = -1 < 0$ and odd degree$= 5$.

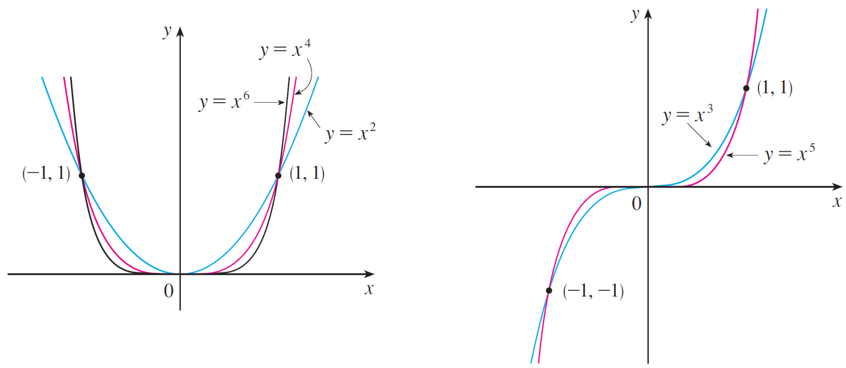
3. Power functions:

$$\boxed{f(x) = x^a}$$

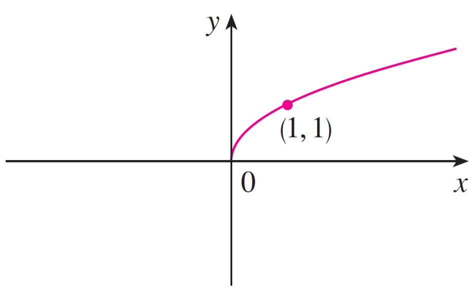
where a is a constant. Here we distinguish three main cases:

(i) $a = n$, where n is a positive integer

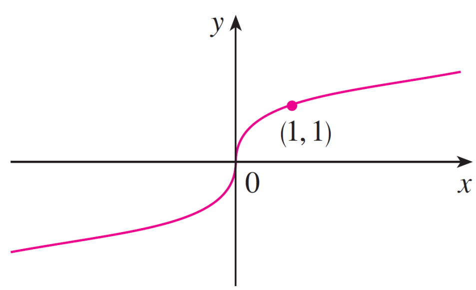




(ii) $a = 1/n$, where n is a positive integer

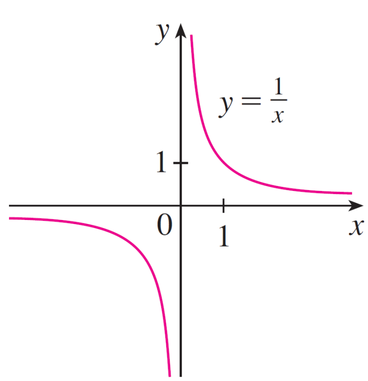


(a) $f(x) = \sqrt{x}$



(b) $f(x) = \sqrt[3]{x}$

(iii) $a = -1$

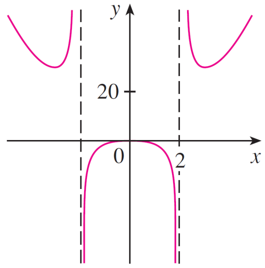


4. Rational functions:

$$\boxed{f(x)=\frac{P(x)}{Q(x)}}$$

where $P(x), Q(x)$ are polynomials.

EXAMPLES: $f(x)=\dfrac{1}{x}$, $g(x)=\dfrac{x+1}{x-3}$, $h(x)=\dfrac{3x^2-5x+1}{x^3+1}$, etc.

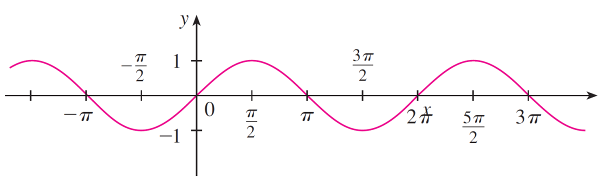


$$f(x)=\frac{2x^4-x^2+1}{x^2-4}$$

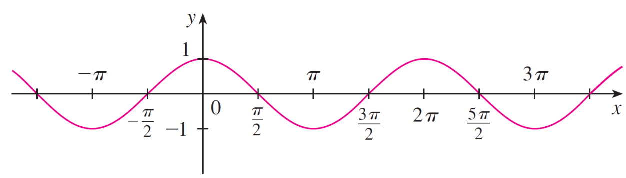
5. Trigonometric functions:

In this course it is important to know graphs and basic properties of the following trigonometric functions:

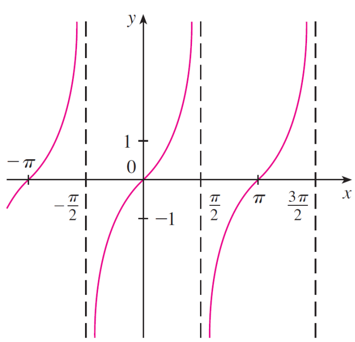
$$\boxed{\sin x,\quad \cos x,\quad \tan x,\quad \cot x,\quad \sec x,\quad \csc x}$$



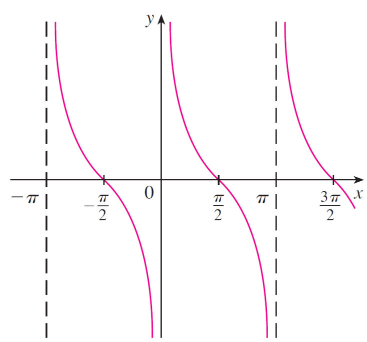
(a) $f(x)=\sin x$



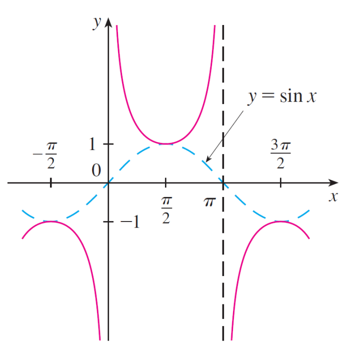
(b) $g(x)=\cos x$



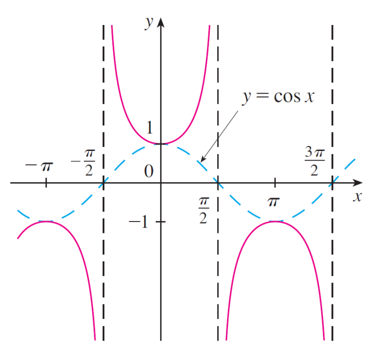
(a) $y=\tan x$



(b) $y=\cot x$



(c) $y=\csc x$

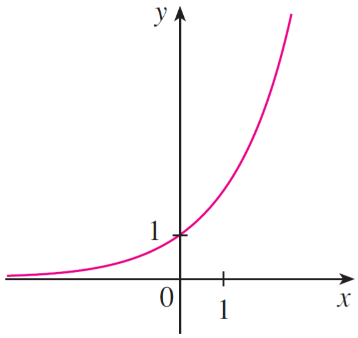


(d) $y=\sec x$

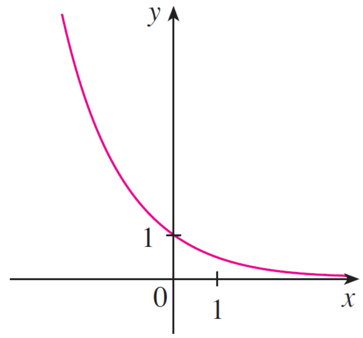
6. Exponential and Logarithmic functions:

$$\boxed{f(x) = a^x, \quad f(x) = \log_a x}$$

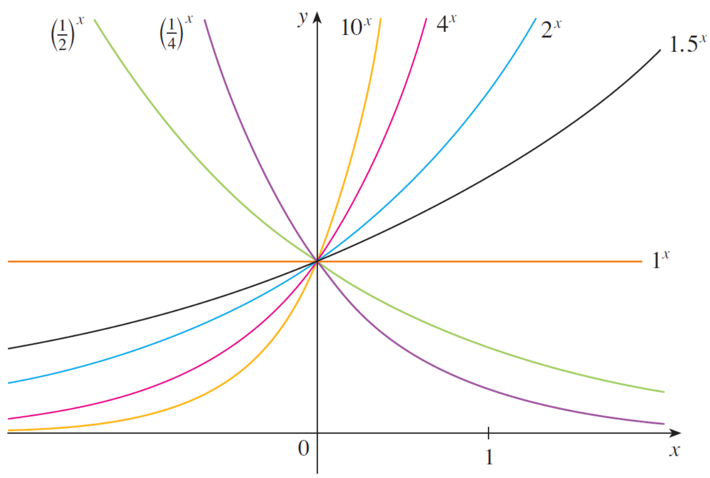
where a is a positive constant.

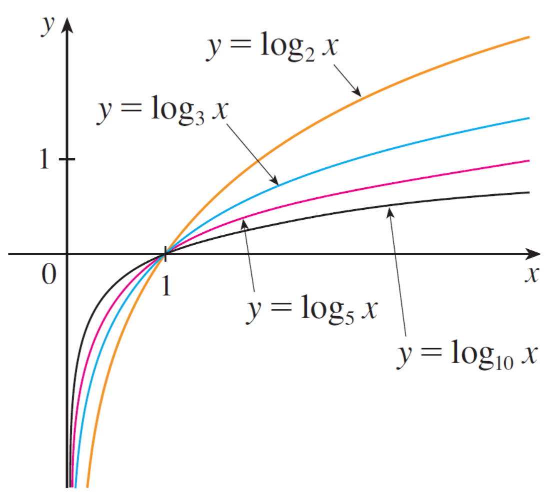


$y = 2^x$



$y = (0.5)^x$





IMPORTANT: Do NOT confuse power functions and exponential functions!

Transformations of Functions

Vertical and Horizontal Shifts: Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left

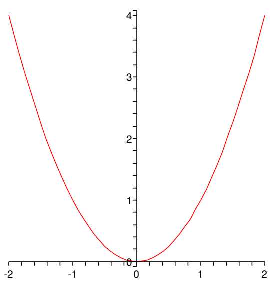
Vertical and Horizontal Stretching and Reflecting: Suppose $c > 0$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, compress the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, compress the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

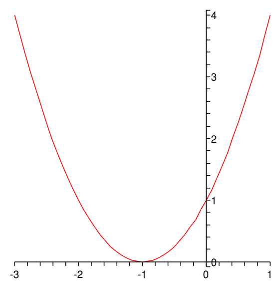
EXAMPLES:

1. Given the graph of $f(x) = x^2$, use transformations to graph $f(x) = (x + 1)^2$.

Step 1: $f(x) = x^2$

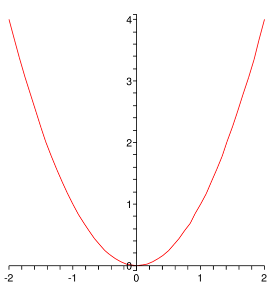


Step 2: $f(x) = (x + 1)^2$ (horizontal shift)

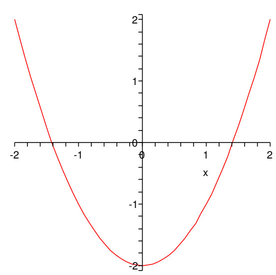


2. Given the graph of $f(x) = x^2$, use transformations to graph $f(x) = x^2 - 2$.

Step 1: $f(x) = x^2$

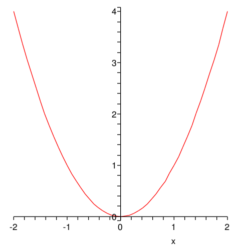


Step 2: $f(x) = x^2 - 2$ (vertical shift)

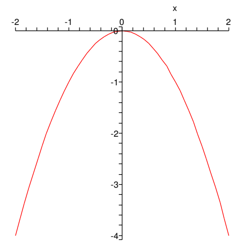


3. Given the graph of $f(x) = x^2$, use transformations to graph $f(x) = -x^2$.

Step 1: $f(x) = x^2$

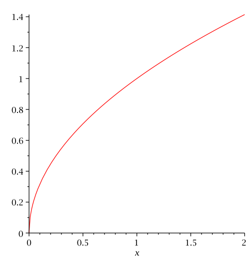


Step 2: $f(x) = -x^2$ (reflection about the x-axis)

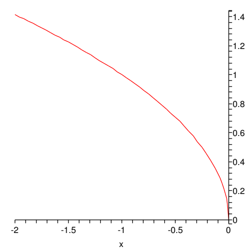


4. Given the graph of $f(x) = \sqrt{x}$, use transformations to graph $f(x) = \sqrt{-x}$.

Step 1: $f(x) = \sqrt{x}$

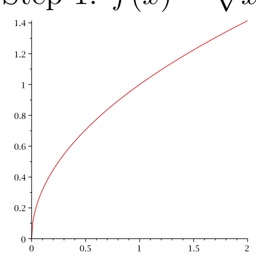


Step 2: $f(x) = \sqrt{-x}$ (reflection about the y-axis)

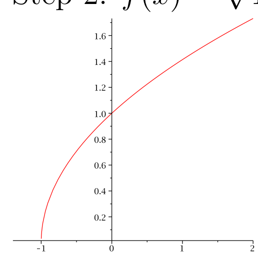


5. Given the graph of $f(x) = \sqrt{x}$, use transformations to graph $f(x) = 1 - \sqrt{1+x}$.

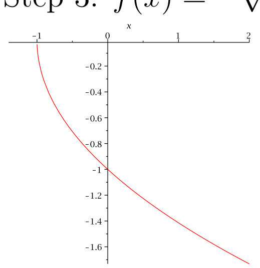
Step 1: $f(x) = \sqrt{x}$



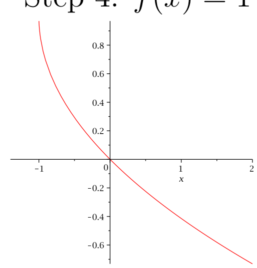
Step 2: $f(x) = \sqrt{1+x}$ (horizontal shift)



Step 3: $f(x) = -\sqrt{1+x}$ (reflection)



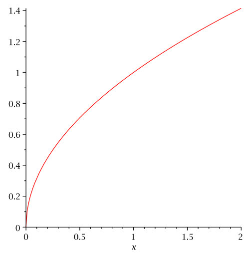
Step 4: $f(x) = 1 - \sqrt{1+x}$ (vertical shift)



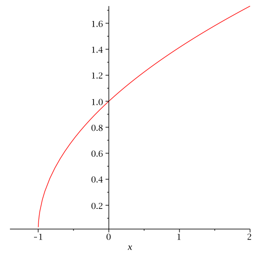
6. Given the graph of $f(x) = \sqrt{x}$, use transformations to graph $f(x) = 1 - \sqrt{1-x}$.

6. Given the graph of $f(x)=\sqrt{x}$, use transformations to graph $f(x)=1-\sqrt{1-x}$.

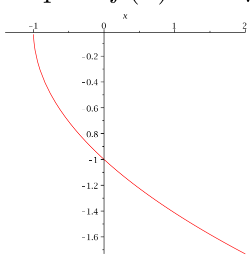
Step 1: $f(x)=\sqrt{x}$



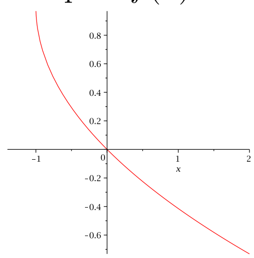
Step 2: $f(x)=\sqrt{1+x}$ (horizontal shift)



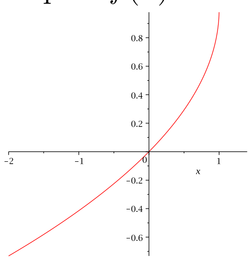
Step 3: $f(x)=-\sqrt{1+x}$ (reflection)



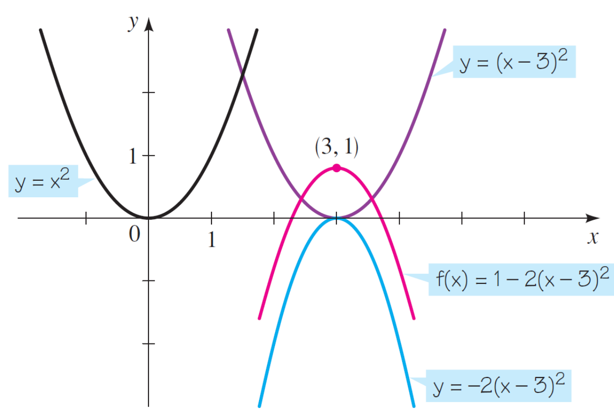
Step 4: $f(x)=1-\sqrt{1+x}$ (vertical shift)



Step 5: $f(x)=1-\sqrt{1-x}$ (reflection about the y-axis)



7. Sketch the graph of the function $f(x)=1-2(x-3)^2$.



Combinations of functions

Two functions f and g can be combined to form new functions $f+g$, $f-g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Algebra of Functions

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

$(f+g)(x) = f(x)+g(x)$	Domain $A \cap B$
$(f-g)(x) = f(x)-g(x)$	Domain $A \cap B$
$(fg)(x) = f(x)g(x)$	Domain $A \cap B$
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$	Domain $\{x \in A \cap B \mid g(x) \neq 0\}$

EXAMPLE: The domain of $f(x)=\sqrt{x}$ is $A=[0,\ \infty)$, the domain of $g(x)=\sqrt{1-x}$ is $B=(-\infty,\ 1]$, and the domain of $h(x)=\sqrt{x-1}$ is $C=[1,\infty)$, so the domain of

$$(f-g)(x)=\sqrt{x}-\sqrt{1-x} \quad \text{is} \quad A\cap B=[0,1]$$

and

$$(f-h)(x)=\sqrt{x}-\sqrt{x-1} \quad \text{is} \quad A\cap C=[1,\infty)$$

EXAMPLE: If $f(x)=x^2$ and $g(x)=x-1$, then the domain of the rational function

$$(f/g)(x)=x^2/(x-1) \quad \text{is} \quad \{x \mid x\neq 1\} \text{ or } (-\infty,\ 1)\cup(1,\ \infty)$$

There is another way of combining two functions to obtain a new function. For example, suppose that $y=f(u)=\sqrt{u}$ and $u=g(x)=x^2+1$. Since y is a function of u and u is, in turn, a function of x, it follows that y is ultimately a function of x. We compute this by substitution:

$$y=f(u)=f(g(x))=f(x^2+1)=\sqrt{x^2+1}$$

The procedure is called *composition* because the new function is *composed* of the two given functions f and g.

Composition of Functions

Given two functions f and g, the **composite function** $f\circ g$ (also called the **composition** of f and g) is defined by

$$(f\circ g)(x) = f(g(x))$$

EXAMPLE: If $f(x)=x^2+1$ and $g(x)=x-3$, find the following.

(a) $f\circ f$ (b) $f\circ g$ (c) $g\circ f$ (d) $g\circ g$ (e) $f(g(2))$ (f) $g(f(2))$

EXAMPLE: If $f(x) = x^2 + 1$ and $g(x) = x - 3$, find the following.

(a) $f \circ f$ (b) $f \circ g$ (c) $g \circ f$ (d) $g \circ g$ (e) $f(g(2))$ (f) $g(f(2))$

Solution: We have

(a) $f \circ f = f(f(x)) = \left\{ \begin{array}{l} f(x^2+1) \\ \text{or} \\ (f(x))^2+1 \end{array} \right\} = (x^2+1)^2+1 = (x^2)^2+2\cdot x^2\cdot 1+1^2+1 = x^4+2x^2+2$

(b) $f \circ g = f(g(x)) = \left\{ \begin{array}{l} f(x-3) \\ \text{or} \\ (g(x))^2+1 \end{array} \right\} = (x-3)^2+1 = x^2-2\cdot x\cdot 3+3^2+1 = x^2-6x+10$

(c) $g \circ f = g(f(x)) = \left\{ \begin{array}{l} g(x^2+1) \\ \text{or} \\ f(x)-3 \end{array} \right\} = (x^2+1)-3 = x^2-2$

(d) $g \circ g = g(g(x)) = \left\{ \begin{array}{l} g(x-3) \\ \text{or} \\ g(x)-3 \end{array} \right\} = (x-3)-3 = x-6$

(e) $f(g(2)) = (2-3)^2+1 = (-1)^2+1 = 1+1 = 2$

(f) $g(f(2)) = 2^2-2 = 4-2 = 2$

EXAMPLE: If $f(x) = x$ and $g(x) = 1$, then

$$f \circ f = x \qquad f \circ g = 1 \qquad g \circ f = 1 \qquad g \circ g = 1$$

REMARK: You can see from the Examples above that sometimes $f \circ g = g \circ f$, but, in general, $f \circ g \neq g \circ f$.

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.

EXAMPLE: If $f(x) = x^2$ and $g(x) = \sqrt{x}$, then

$$f \circ f = (x^2)^2 = x^4 \qquad f \circ g = x,\ x \geq 0 \qquad g \circ f = |x| \qquad g \circ g = \sqrt{\sqrt{x}} = \sqrt[4]{x}$$

(of course, the domain of $g \circ g = \sqrt[4]{x}$ is all nonnegative numbers).

EXAMPLE: If $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$, then

$$f \circ f = (x^3)^3 = x^9 \qquad f \circ g = x \qquad g \circ f = x \qquad g \circ g = \sqrt[3]{\sqrt[3]{x}} = \sqrt[9]{x}$$

EXAMPLE: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each function and its domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

EXAMPLE: If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{2-x}$, find each function and its domain.

(a) $f\circ g$ (b) $g\circ f$ (c) $f\circ f$ (d) $g\circ g$

Solution:

(a) We have

$$(f\circ g)(x)=f(g(x))=f(\sqrt{2-x})=\sqrt{\sqrt{2-x}}=\sqrt[4]{2-x}$$

The domain of $f\circ g$ is $\{x \mid 2-x\geq 0\}=\{x \mid x\leq 2\}=(-\infty,\ 2]$.

(b) We have

$$(g\circ f)(x)=g(f(x))=g(\sqrt{x})=\sqrt{2-\sqrt{x}}$$

For $\sqrt{x}$ to be defined we must have $x\geq 0$. For $\sqrt{2-\sqrt{x}}$ to be defined we must have $2-\sqrt{x}\geq 0$, that is, $\sqrt{x}\leq 2$, or $x\leq 4$. Thus we have $0\leq x\leq 4$, so the domain of $g\circ f$ is the closed interval $[0,4]$.

(c) We have

$$(f\circ f)(x)=f(f(x))=f(\sqrt{x})=\sqrt{\sqrt{x}}=\sqrt[4]{x}$$

The domain of $f\circ f$ is $[0,\ \infty)$.

(d) We have

$$(g\circ g)(x)=g(g(x))=g(\sqrt{2-x})=\sqrt{2-\sqrt{2-x}}$$

This expression is defined when both $2-x\geq 0$ and $2-\sqrt{2-x}\geq 0$. The first inequality means $x\leq 2$, and the second is equivalent to $\sqrt{2-x}\leq 2$, or $2-x\leq 4$, or $x\geq -2$. Thus $-2\leq x\leq 2$, so the domain of $g\circ g$ is the closed interval [-2, 2].

It is possible to take the composition of three or more functions. For instance, the composite function $f\circ g\circ h$ is found by first applying h, then g, and then f as follows:

$$(f\circ g\circ h)(x)=f(g(h(x)))$$

EXAMPLE: Find $f\circ g\circ h$ if $f(x)=x/(x+1)$, $g(x)=x^{10}$, and $h(x)=x+3$.

Solution: We have

$$(f\circ g\circ h)(x)=f(g(h(x)))=f(g(x+3))=f((x+3)^{10})=\frac{(x+3)^{10}}{(x+3)^{10}+1}$$

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE: Given $F(x)=\dfrac{1}{(x+9)^2}$, find functions f,g, and h such that $F=f\circ g\circ h$.

EXAMPLE: Given $F(x) = \dfrac{1}{(x+9)^2}$, find functions f, g, and h such that $F = f \circ g \circ h$.

Solution 1: The formula for F says: First add 9, then square $x+9$, and finally divide 1 by the result. So we let

$$f(x) = \frac{1}{x}, \quad g(x) = x^2, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+9)) = f((x+9)^2) = \frac{1}{(x+9)^2} = F(x)$$

Solution 2: Here is an other way to look at F: First add 9, then divide 1 by $x+9$, and finally square the result. So we let

$$f(x) = x^2, \quad g(x) = \frac{1}{x}, \quad h(x) = x + 9$$

Then

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+9)) = f\left(\frac{1}{x+9}\right) = \left(\frac{1}{x+9}\right)^2 = \frac{1}{(x+9)^2} = F(x)$$