Functions and their Representations

DEFINITION: A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. It's **graph** is the set of ordered pairs

$$\{(x, f(x)) \mid x \in A\}$$

The set A is called the **domain** of f. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

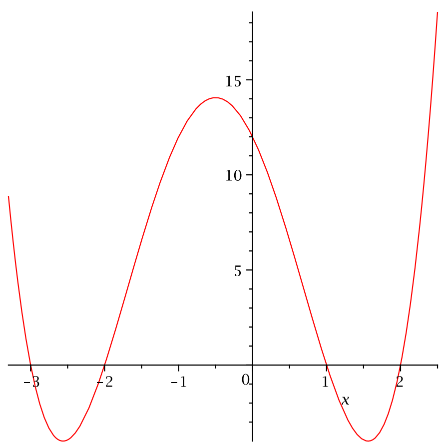
There are 4 possible ways to represent a function:

- Verbally (in words)
- Visually (by a graph)
- Numerically (by a table of values)
- Algebraically (by an explicit formula)

EXAMPLES:

1. Verbally: **"$s(t)$ is speed of a car at time t"**

2. Visually:



3. Numerically:

x	$f(x)$
1	2
2	8
7	-1
10	5

4. Algebraically:

(i) $f(x) = 1,\ g(x) = \dfrac{x}{x},\ h(x) = x^2,\ F(x) = \dfrac{1}{x},\ y = \dfrac{5\sin x - 4}{\ln x}$

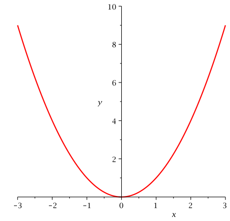
(ii) $f(x) = \dfrac{1}{x},\ x > 5$

(iii) $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ x+1 & \text{if } x > 2 \end{cases}$

Domain and Range

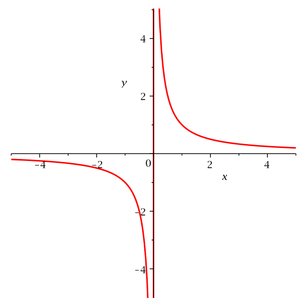
EXAMPLES:

1. $f(x) = x^2$



Domain: All real numbers or $(-\infty, \infty)$.
Range: $\{y \mid y \geq 0\}$ or $[0, \infty)$.

2. $f(x) = \dfrac{1}{x}$



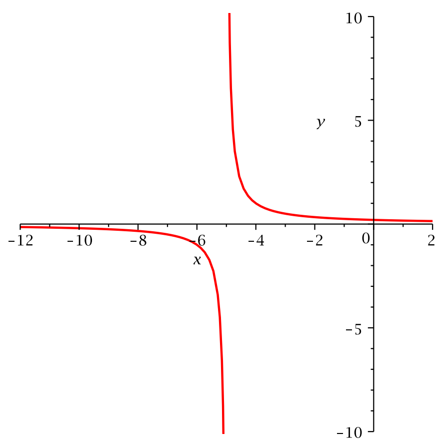
Domain: $\{x \mid x \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$, since $x \neq 0$.
Range: $\{y \mid y \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$.

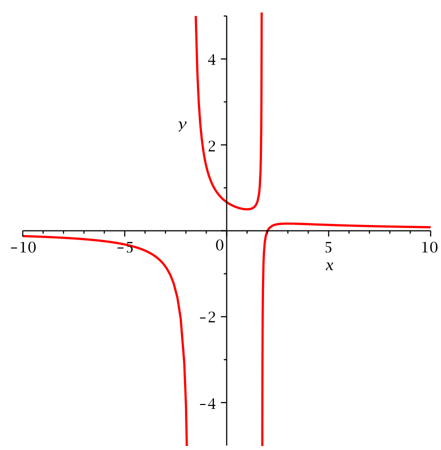
3. $f(x) = \dfrac{1}{x}$, $x \geq 2$

Domain: $\{x \mid x \geq 2\}$ or $[2, \infty)$, since $x \geq 2$.
Range: $\{y \mid 0 < y \leq 1/2\}$ or $(0, 1/2]$.

4. $f(x) = \dfrac{1}{x+5}$

Domain: $\{x \mid x \neq -5\}$ or $(-\infty, -5) \cup (-5, \infty)$, since $x + 5 \neq 0$.
Range: $\{y \mid y \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$.





5. $f(x) = \dfrac{x-2}{x^2-3}$

Domain: $\{x \mid x \neq \pm\sqrt{3}\}$ or $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$, since $x^2 - 3 \neq 0$.
Range: $\left(-\infty, \dfrac{1}{6}\right] \cup \left[\dfrac{1}{2}, \infty\right)$.

6. $f(x) = \sqrt{2-x}$

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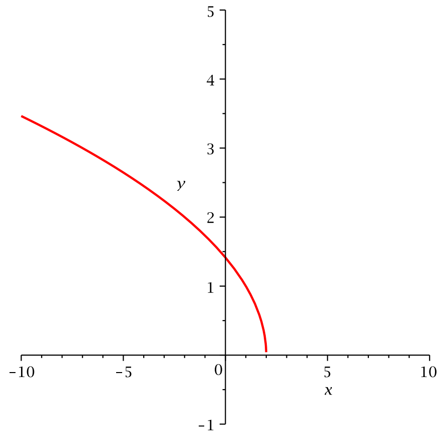
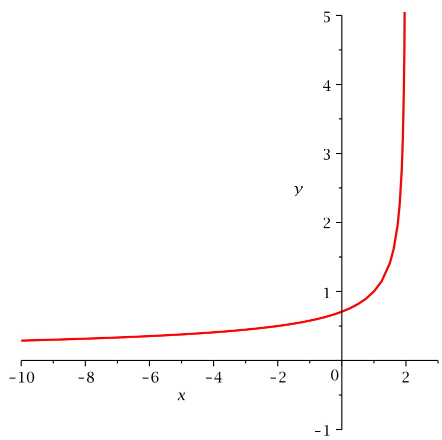
Domain: $\{x \mid x \le 2\}$ or $(-\infty, 2]$, since $2 - x \ge 0$.
Range: $\{y \mid y \ge 0\}$ or $[0, \infty)$.



7. $f(x) = \dfrac{1}{\sqrt{2-x}}$

Domain: $\{x \mid x < 2\}$ or $(-\infty, 2)$, since $2 - x > 0$.
Range: $\{y \mid y > 0\}$ or $(0, \infty)$.



8. $f(x) = \dfrac{1}{\sqrt{2x - x^2}}$

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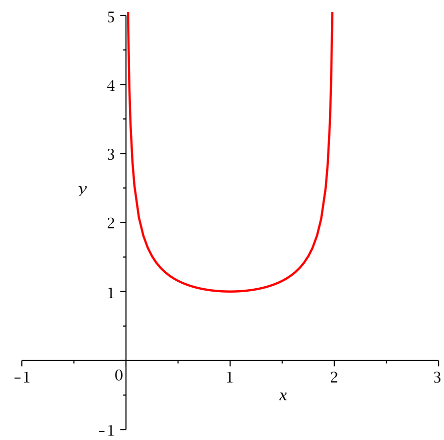
Domain: $\{x \mid 0 < x < 2\}$ or $(0, 2)$, since $2x - x^2 = x(2 - x) > 0$.
Range: $\{y \mid y \geq 1\}$ or $[1, \infty)$.

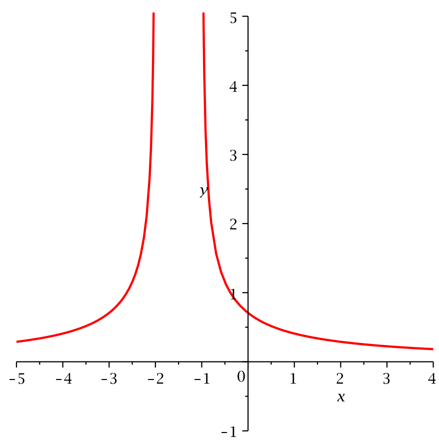


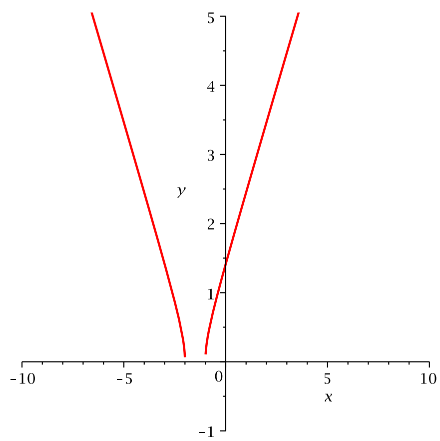
9. $f(x) = \dfrac{1}{\sqrt{x^2 + 3x + 2}}$

Domain: $\{x \mid x < -2 \text{ or } x > -1\}$ or $(-\infty, -2) \cup (-1, \infty)$, since $x^2 + 3x + 2 = (x+1)(x+2) > 0$.
Range: $\{y \mid y > 0\}$ or $(0, \infty)$.

10. $f(x) = \sqrt{x^2 + 3x + 2}$

Domain: $\{x \mid x \leq -2 \text{ or } x \geq -1\}$ or $(-\infty, -2] \cup [-1, \infty)$, since $x^2 + 3x + 2 = (x+1)(x+2) \geq 0$.
Range: $\{y \mid y \geq 0\}$ or $[0, \infty)$.

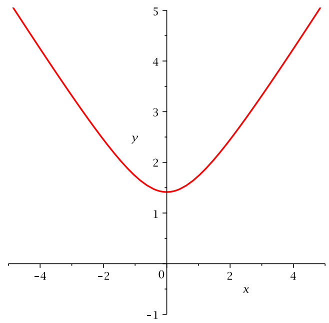




11. $f(x) = \sqrt{x^2 + 2}$

11. $f(x) = \sqrt{x^2 + 2}$

Domain: All real numbers, or $(-\infty, \infty)$, since $x^2 + 2$ is always > 0.
Range: $\{y \mid y \geq \sqrt{2}\}$ or $[\sqrt{2}, \infty)$.

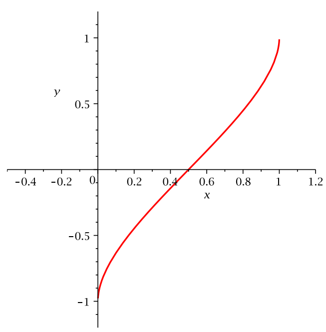


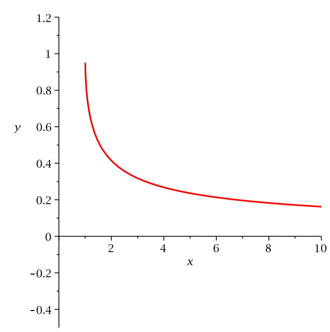
12. $f(x) = \sqrt{x} - \sqrt{1 - x}$

Domain: $\{x \mid 0 \leq x \leq 1\}$ or $[0, 1]$, since $x \geq 0$ and $1 - x \geq 0$.
Range: $\{y \mid -1 \leq y \leq 1\}$ or $[-1, 1]$.

13. $f(x) = \sqrt{x} - \sqrt{x - 1}$

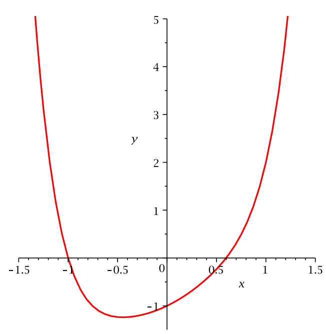
Domain: $\{x \mid x \geq 1\}$ or $[1, \infty)$, since $x \geq 0$ and $x - 1 \geq 0$.
Range: $\{y \mid 0 < y \leq 1\}$ or $(0, 1]$.





14. $f(x) = x^6 + x^2 + x - 1$

Domain: $(-\infty, \infty)$.
Range: $\{y \mid y \geq -1.2392\}$ or $[-1.2392, \infty)$.



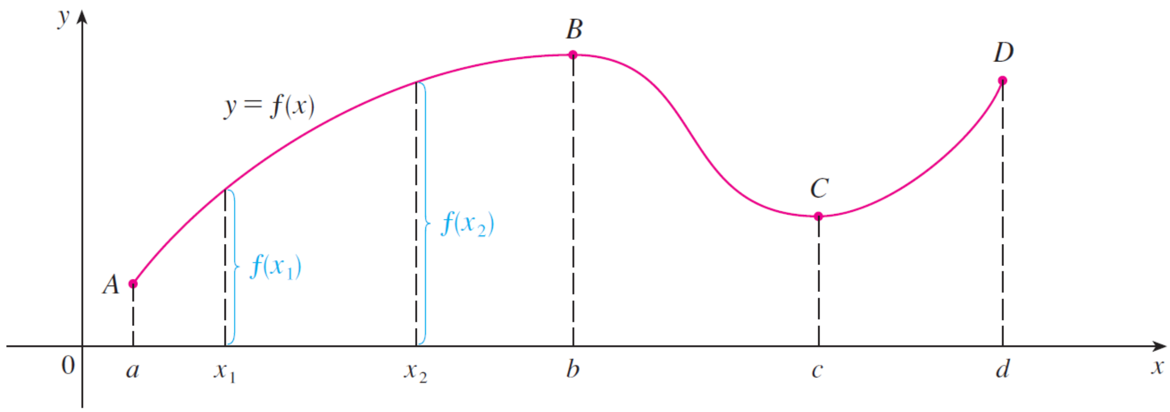
Increasing and Decreasing Functions

DEFINITION: A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

It is called **decreasing** on an I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$



EXAMPLES:

1. The function $f(x) = x^2$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.
2. The function $f(x) = x^3$ is increasing everywhere, that is on $(-\infty, \infty)$.
3. The function $f(x) = \dfrac{1}{x}$ is decreasing on $(-\infty, 0)$ and on $(0, \infty)$.

Even and odd functions

DEFINITION: A function f that satisfies

$$\boxed{f(-x) = f(x)}$$

for every number x in its domain is called an **even function**. A function f that satisfies

$$\boxed{f(-x) = -f(x)}$$

for every number x in its domain is called an **odd function**.

REMARK: Any function is either even, or odd, or neither.

PROPERTY: Graphs of even functions are symmetric with respect to the y-axis. Graphs of odd functions are symmetric with respect to the origin.

IMPORTANT: Do NOT confuse even/odd functions and even/odd integers!

EXAMPLES:

1. Functions $f(x) = x^2,\ x^4,\ x^8,\ x^4 - x^2, x^2 + 1, |x|, \cos x$, etc. are even. In fact,
   - if $f(x) = x^2$, then $f(-x) = (-x)^2 = x^2 = f(x)$
   - if $f(x) = x^4$, then $f(-x) = (-x)^4 = x^4 = f(x)$
   - if $f(x) = x^8$, then $f(-x) = (-x)^8 = x^8 = f(x)$
   - if $f(x) = x^4 - x^2$, then $f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x)$
   - if $f(x) = x^2 + 1$, then $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$
   - if $f(x) = |x|$, then $f(-x) = |-x| = |x| = f(x)$
   - if $f(x) = \cos x$, then $f(-x) = \cos(-x) = \cos x = f(x)$

One can see that graphs of all these functions are symmetric with respect to the y-axis.

2. Functions $f(x) = x,\ x^3,\ x^5,\ x^3 - x^7, \sin x$, etc. are odd. In fact,
   - if $f(x) = x$, then $f(-x) = -x = -f(x)$
   - if $f(x) = x^3$, then $f(-x) = (-x)^3 = -x^3 = -f(x)$
   - if $f(x) = x^5$, then $f(-x) = (-x)^5 = -x^5 = -f(x)$
   - if $f(x) = x^3 - x^7$, then $f(-x) = (-x)^3 - (-x)^7 = -x^3 + x^7 = -(x^3 - x^7) = -f(x)$
   - if $f(x) = \sin x$, then $f(-x) = \sin(-x) = -\sin x = -f(x)$

One can see that graphs of all these functions are symmetric with respect to the origin.

3. Functions $f(x) = x + 1,\ x^3 + x^2,\ x^5 - 2,\ |x - 2|$ etc. are neither even nor odd. In fact,
   - if $f(x) = x + 1$, then $f(-1) = -1 + 1 = 0,\ f(1) = 1 + 1 = 2$, therefore $f(-1) \neq \pm f(1)$
   - if $f(x) = x^3 + x^2$, then $f(-1) = (-1)^3 + (-1)^2 = -1 + 1 = 0,\ f(1) = 1^3 + 1^2 = 2$, therefore $f(-1) \neq \pm f(1)$
   - if $f(x) = x^5 - 2$, then $f(-1) = (-1)^5 - 2 = -1 - 2 = -3,\ f(1) = 1^5 - 2 = 1 - 2 = -1$, therefore $f(-1) \neq \pm f(1)$
   - if $f(x) = |x - 2|$, then $f(-1) = |-1 - 2| = |-3| = 3,\ f(1) = |1 - 2| = |-1| = 1$, therefore $f(-1) \neq \pm f(1)$